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The Lefschetz Fixed-Point Formula; Smoothness and Stability

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It is a great honor and pleasure for me to be speaking at the formal occasion for dedicating the Brown University Center for Dynamical Systems to the memory of Solomon Lefschetz. Lefschetz is surely one of the towering figures of mathematics in our century, and while he is mainly thought of for his contributions to algebraic geometry and topology, his contributions to dynamical systems have been great. Let me mention just two that are of special interest to me and that I shall be talking about today. First, there is the Lefschetz fixed-point formula. While in some sense this formula has become part of topology, topology used to be called analysis situs. As Dennis Sullivan put it to me one day: "Topology is the subject; analysis is the object." This, of course, is a major theme from Poincaré to the present. Lefschetz never lost sight of it. With respect to the fixed-point formula one need only read Lefschetz's discussion of it in the preface to [13] or in the introduction to [14]. In the latter, Lefschetz refers to Poincaré's interest in fixed points, which came out of dynamics. In case a vector field X has a manifold of section M, the periodic solutions of X correspond to the periodic points of the first return map on M. This is, of course, but one simple application of this very general and powerful formula. Second, I would like to mention structural stability. Structural stability was defined by Andronov and Pontryagin in 1937 [2] and they stated results on structural stability in two dimensions. While I know of no work of Lefschetz himself on structural stability we can find his influence quite clearly stated in the works of DeBaggis [6] and Peixoto [22], where some of the fundamental theorems of structural stability were first proven.

I. The Lefschetz Fixed-Point Formula and Smoothness

The Lefschetz fixed-point formula is known for quite general spaces (e.g., finite simplicial complexes) but I will restrict my discussion here to compact differentiable manifolds without boundary.

If 0 is an isolated fixed point of the continuous map $f\colon U \to R^m$, where U is an open subset of R^m, then the index of f at 0, $\sigma_f(0)$, is the local degree of the mapping $Id - f$ restricted to an appropriately small open set about 0. If 0 is an isolated fixed point of f^n, then $\sigma_{f^n}(0)$ is defined for all n. If $f\colon M \to M$, $f^n(p) = p$, and p is an isolated fixed point for f^n, then $\sigma_{f^n}(p)$ is defined by taking local coordinates around p. The Lefschetz numbers $L(f^n)$ are defined by the formula

$$L(f^n) \equiv \sum (-1)^i \operatorname{tr} f^n_{*i}\colon H_i(M) \to H_i(M),$$

where $H_i(M)$ is the ith homology group of M with rational coefficients and $f^n_{*i}\colon H_i(M) \to H_i(M)$ the map induced by f on these groups. $(f^n)_{*i} = (f_{*i})^n$, so it is not too difficult to compute the $L(f^n)$ once the eigenvalues of the f_{*i} are known.

The Lefschetz fixed-point formula says that the $L(f^n)$ can be computed in terms of the fixed-point indices.

(1.1) Lefschetz fixed-point theorem

$$L(f^n) = \sum_{p \in \operatorname{Fix} f^n} \sigma_{f^n}(p)$$

provided that the fixed points of f^n are isolated.

Of course, this theorem has the immediate corollary that f has a fixed point if $L(f) \neq 0$; but it says much more. Let me give an example of the type that I shall be pursuing. Let $0 < r < \infty$ and $E^r(M)$ denote the space of C^r endomorphisms $f\colon M \to M$ with the C^r-topology. $E^r(M)$ is a complete metric space, and hence the Baire category theorem applies. A subset $X \subset E^r(M)$ is said to be generic if it contains the countable intersection of open and dense sets.

By the Baire category theorem a generic set in $E^r(M)$ is dense. If a property is true for a generic subset of $E^r(M)$ we say that it is generic or that the generic f has that property. The simplest part of the Kupka–Smale theorem (see [27], for example) shows that the periodic points of period n of the generic f (i.e., the fixed points of f^n) are transversal, hence isolated, and the index of any periodic point of generic f is either $+1$ or -1. Let $N_n(f)$ be the number of period points of period n of f.

(1.2) Proposition. For the generic f in $E^r(M)$,

$$N_n(f) \geq |L(f^n)|.$$

This is an immediate consequence of the Lefschetz fixed-point formula.

Example 1. For the generic C^r, $f\colon S^m \to S^m$,

$$N_n(f) \geq |\deg(f)^n \pm 1|$$

where the sign is positive if m is even and negative if m is odd.

These formulas are valid for all n and thus we can try to summarize the information contained in them by a single number that measures the asymptotic exponential growth rate of the $N_n(f)$.

(1.3) Proposition. For the generic f in $E^r(M)$,

$$\limsup (1/n) \log N_n(f) \geq \limsup (1/n) \log |L(f^n)|.$$

Another way to express the number $\limsup (1/n) \log N_n(f)$ is that it is the reciprocal of the radius of convergence of the Artin–Mazur zeta function

$$\zeta_f(t) = \exp\left[\sum_{n=1}^{\infty} N_n(f)t^n/n\right].$$

Neither Proposition 1 nor 2 holds for all continuous, Lipschitz, or even piecewise linear mappings.

Example 2. Let $f\colon S^2 \to S^2$ be the map of the Riemann sphere defined by $z \to 2z^2/\|z\|$. Then $\deg(f) = 2$, but f has only two periodic points 0 and ∞. So $N_n(f) = 2$ for all n, whereas $L(f^n) = 2^n + 1$. This f is Lipschitz but not piecewise linear. There are piecewise linear homeomorphisms of manifolds even that do not satisfy this property. The Lefschetz formula is valid, of course, in the case $\sigma_{f^n}(0) + \sigma_{f^n}(\infty) = 2^n + 1$. It is not difficult to calculate that $\sigma_{f^n}(0) = 2^n$.

Let me digress for a moment to emphasize the power of formulas like Proposition (1.3). In principal if we are to know or even to estimate the growth rate of the number of periodic points of f of period n for all n, we must know f precisely and we must iterate it infinitely often. On the other hand, $\limsup (1/n) \log |L(f^n)|$ can be calculated. If we know the manifold M and if we know f even approximately, then by simplicial approximation the eigenvalues of the f_{*i} can be calculated, and $\limsup (1/n) \log |L(f^n)|$ can be determined. If, for example, the f_{*i} have a unique eigenvalue of maximal modulus, λ, then $\limsup (1/n) \log |L(f^n)| = \log |\lambda|$. The total effect is to replace a nonlinear problem involving infinite iterations by a finite combinatorial problem and to make it linear.

Example 2 shows that Propositions (1.2) and (1.3) fail for C^0 endomorphisms of manifolds. In fact, there are only two periodic points, while the Lefschetz numbers tend to infinity. But surprisingly enough it is not known if Proposition (1.3) is true for smooth maps.

Problem 1 [34]. Let $f\colon M \to M$ be smooth. Is $\limsup (1/n) \log N_n(f) \geq \limsup (1/n) \log |L(f^n)|$?

It is only recently that effects of smoothness on the Lefschetz formula have begun to be studied.

(1.4) Proposition [34]. Suppose $f\colon U \to R^m$ is C^1 and that 0 is an isolated fixed point of f^n for all n. Then $\sigma_{f^n}(0)$ is bounded as a function of n.

The Lefschetz trace formula now tells us:

(1.5) Corollary [34]. If $f\colon M \to M$ is C^1 and the Lefschetz numbers $L(f^n)$ are not bounded then the set of periodic points of f is infinite.

So the phenomenon of Example 2 cannot occur for C^1 mappings. Any C^1 mapping $f\colon S^2 \to S^2$ with degree 2 has infinitely many periodic points.

(1.5′) Corollary. Suppose the C^1 vector field X on N has a C^1 manifold of section M, with first return map $f\colon M \to M$. If the Lefschetz numbers $L(f^n)$ are unbounded, then X has infinitely many periodic solutions.

Example 3. To illustrate Corollaries (1.5) and (1.5′) we consider what has come to be called the Thom diffeomorphism of the two-dimensional torus T^2. Think of T^2 as R^2 mod the integer lattice $Z^2 \subset R^2$. $T^2 = R^2/Z^2$. The matrix $A = \binom{2\,1}{1\,1}$ has integer entries and determinant one, so A defines a map of R^2/Z^2 that is invertible, i.e., a diffeomorphism. Denote the map of T^2 it defines by A again.

$A\colon T^2 \to T^2$. The homology groups of T^2 with real coefficients are

$$H_0(T^2, R) = R, \qquad H_1(T^2, R) = R^2, \qquad H_2(T^2, R) = R.$$

A_{0*} and A_{1*} are the identity transformations and $A_{1*}\colon R^2 \to R^2$ is just A itself. Thus the Lefschetz numbers $L(A^n) = 2 - \operatorname{tr} A^n$. The eigenvalues of A are $(3 \pm \sqrt{5})/2$. So

(1) The $L(A^n)$ are unbounded; in fact,
(2) $\limsup (1/n) \log |L(A^n)| = \log [(3 + \sqrt{5})/2]$.

Consequently:

(a) Any $C^1 f$ homotopic to A has infinitely many periodic points.

(b) Any C^1 vector field X that has a C^1, T^2 manifold of section with first return map homotopic to A has infinitely many periodic solutions.

(c) The generic f in $E^r(T^2)$ that is homotopic to A has

$$\limsup\, (1/n) \log N_n(f) \geq \log [(3 + \sqrt{5})/2],$$

so the number of periodic points of f of period n is growing exponentially with n.

II. Structural Stability

The theory of structural stability (and Ω-stability) for n-dimensional manifolds has made remarkable progress in the last fifteen years. We recall these concepts briefly.

(2.1) Definition. If X and Y are topological spaces, $f: X \to X$ and $g: Y \to Y$ are continuous, then f and g are topologically conjugate iff there is a surjective homeomorphism $h: X \to Y$ such that $gh = hf$.

(2.2) Definition. If X is a topological space and $f: X \to X$ is continuous, the nonwandering set of f, $\Omega(f) = \{\psi \in X \mid$ given any neighborhood U_ψ in X there is an $n > 0$ such that $f^n(U_\psi) \cap U_\psi \neq \varnothing\}$.

$\Omega(f)$ is closed, $f(\Omega(f)) \subset \Omega(f)$, and if f is a homeomorphism $f(\Omega(f)) = \Omega(f)$ and $\Omega(f) = \Omega(f^{-1})$. $\Omega(f)$ contains all the periodic points of f and all the ω-limit points of f. So as Smale has said $\Omega(f)$ is where all the action is.

(2.3) Definition. (a) $f \in \mathrm{Diff}^r(M)$ is structurally stable if there is a neighborhood V_f of f in $\mathrm{Diff}^r(M)$ such that any $g \in V_f$ is topologically conjugate to f.

(b) $f \in \mathrm{Diff}^r(M)$ is Ω-stable if there is a neighborhood V_f of f in $\mathrm{Diff}^r(M)$ such that for any $g \in V_f$, $f/\Omega(f)$ and $g/\Omega(g)$ are topologically conjugate.

Any structurally stable diffeomorphism is Ω-stable, but there are Ω-stable diffeomorphisms that are not structurally stable. This was proven by Smale, which is why he defined Ω-stability. For much more discussion of structural and Ω-stability and examples of these phenomena one should read Smale's original survey article on the subject [39], and one should consult Nitecki's book [18] as well as the further survey articles [29, 40, 42]. Lefschetz has perhaps made the strongest claim for the importance of structural stability itself [15, p. 250]:

> In a system of differential equations arising out of a practical problem the coefficients are only known approximately: various errors are inevitably involved in their determination. The phase portrait of the system must therefore be such that it is not affected by small modifications in the coefficients. In other words it must be "stable" under these conditions.

Lefschetz was, of course, talking about the definition of structural stability for differential equations, but the same may be said for diffeomorphisms. Insisting that the systems must be stable in the sense of structurally stable is perhaps extreme. Other authors, for example Thom, content themselves with less. Lefschetz was perhaps influenced by the situation for two manifolds that he was talking about, but in any case his statement gives the flavor of the importance of structurally stable systems, and much the same can be said for Ω-stable systems. Moreover, while it is to be expected that structurally stable diffeomorphisms exhibit the generic behavior of nearby systems, they will not have topological properties that are not generic. Thus structurally stable diffeomorphisms avoid pathological behavior and should be fairly simple to describe. Since $\mathrm{Diff}^r(M)$ is separable, there are only countably many structurally stable systems up to topological conjugacy, and one might hope to find invariants that give a fairly accurate picture of the orbit structure of structurally stable systems.

To give a brief example, if $f \in \mathrm{Diff}^r(M)$ is Ω-stable then $N_n(f) = N_n(g)$ for all n and all g in some C^r neighborhood of f. If we now apply Proposition (1.3) we have

(2.4) Proposition. If $f \in \mathrm{Diff}^r(M)$ is Ω-stable, then

$$\limsup (1/n) \log N_n(f) \geq \limsup (1/n) \log |L(f^n)|.$$

Now we are faced with an alternative: either there is no diffeomorphism $f\colon T^2 \to T^2$ that is even homologous to the Thom diffeomorphism (see Example 3) and is structurally stable or there are structurally stable diffeomorphisms with an infinite number of periodic points. This was the motivation for Thom's example, and with historical hindsight the situation is clear enough. But at the time it was thought that the structurally stable diffeomorphism might be open and dense on all manifolds and would have only finitely many periodic points. The reason for this is that Peixoto had proven the openness and density of structurally stable diffeomorphisms in $\mathrm{Diff}^r(S^1)$ and had shown that they had only finitely many periodic points.

Actually, Smale constructed a structurally stable diffeomorphism of S^2 with infinitely many periodic points [37], and Anosov [3] later proved that the Thom diffeomorphism was structurally stable. The difference between S^1, S^2, and T^2 can be seen in the following remark. If $f\colon S^m \to S^m$ is a diffeomorphism, then $|L(f^n)| = 0$ or 2 for all n.

Thus Thom's example shows that infinitely many periodic points can be forced for homological reasons, because of the Lefschetz fixed-point formula. Smale's example shows that they are not pathological and that they occur in structurally stable diffeomorphisms for local as well as global topological reasons. I have dwelt on the Thom diffeomorphism because I will investigate below further homological restrictions on the growth rate of the number of periodic points of the Ω-stable diffeomorphisms that are understood. But first, I have to present the picture that evolved through the work of Smale and others describing Ω-stable and structurally stable diffeomorphisms.

(2.5) **Axiom A** [39]. $f \in \mathrm{Diff}^r(M)$ satisfies Smale's Axiom A if and only if

(a) $\Omega(f)$ has a hyperbolic structure,
(b) $\Omega(f)$ is the closure of the periodic points of f.

That $\Omega(f)$ has a hyperbolic structure means that $TM|\Omega(f)$, the tangent bundle of M restricted to $\Omega(f)$, may be written as the direct sum of two Tf invariant subbundles $E^s \oplus E^u$ such that there exist constants $0 < \lambda < 1$, $0 < c$, and

$$\|Tf^n|E^s\| \le c\lambda^n \quad \text{for} \quad n > 0,$$
$$\|Tf^n|E^u\| \le c\lambda^n \quad \text{for} \quad n < 0.$$

When f satisfies Axiom A, Smale proved the existence of a "spectral decomposition" for $\Omega(f)$.

(2.6) **Theorem** [39]. If $f \in \mathrm{Diff}^r(M)$ satisfies Axiom A, then $\Omega(f)$ is the disjoint union $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$, where each Ω_i is closed and invariant for f and $f|\Omega_i$ is topologically transitive.

(2.7) Corollary [39]. If $f\colon M \to M$ is as above, then M is the disjoint union of

(a) $M = \bigcup_{i=1}^{k} W^s(\Omega_i)$

where

$$W^s(\Omega_i) = \{\psi \in M \mid f^m(\psi) \to \Omega_i \text{ as } m \to \infty\}.$$

(b) $M = \bigcup_{i=1}^{k} W^u(\Omega_i)$

where

$$W^u(\Omega_i) = \{\psi \in M \mid f^m(\psi) \to \Omega \text{ as } m \to -\infty\}.$$

Under the circumstances we can define $\Omega_i > \Omega_j$ if $(W^u(\Omega_i) - \Omega_i) \cap (W^s(\Omega_j) - \Omega_j) \neq \varnothing$. f is said to have no cycles if $\Omega_{i_0} > \Omega_{i_1} > \cdots > \Omega_{i_j} = \Omega_{i_0}$ is impossible for any $j \geq 1$. Actually it is always impossible for $\Omega_i > \Omega_i$ for an Axiom A diffeomorphism as Smale knew, so we can say $j \geq 2$ if we want.

We can finally state Smale's Ω stability theorem:

(2.8) Theorem (Smale [41]). If f satisfies Axiom A and has no cycles then f is Ω-stable.

The converse to this theorem, which was conjectured by Smale, is one of the outstanding problems of dynamical systems.

(2.9) Conjecture (Smale [41, 42]). f is Ω-stable iff f satisfies Axiom A and the no-cycle property.

It is known that

(2.10) Theorem (Palis [20]). If f satisfies Axiom A and is Ω-stable then f has no cycles.

So the problem really is: Does Ω-stability imply Axiom A? There are some partial results here by Franks [7, 8], Guckenheimer [9], and more recently Mañé [16], but the problem is still open. Moreover, all these approaches are strictly C^1 because they rely on Pugh's general density theorem, which states:

(2.11) Theorem (Pugh [24]). For the generic f in $\mathrm{Diff}^1(M)$, $\Omega(f) = \overline{\mathrm{per}(f)}$.

So anything that is Ω-stable in $\mathrm{Diff}^1(M)$ already satisfies Axiom A(b).

To know whether Pugh's closing lemma or general density theorem holds in $\mathrm{Diff}^r(M)$, $r > 1$, is absolutely crucial for our thinking. Personally, I find the C^1 proof so difficult already that I frequently despair and try to find counterexamples.

Turning our attention to structural stability we define the strong transversality condition as in Smale [40]. If $f \in \mathrm{Diff}^r(M)$ and $\psi \in M$,

$$W^s(\psi) = \{y \in M \mid d(f^n(\psi), f^n(y)) \to 0 \text{ as } n \to \infty\}$$
$$W^u(\psi) = \{y \in M \mid d(f^n(\psi), f^n(y)) \to 0 \text{ as } n \to -\infty\}.$$

If f satisfies Axiom A, it follows [10, 39] that $W^s(\psi)$ and $W^u(\psi)$ are $1:1$ immersed Euclidean spaces for all $\psi \in M$.

(2.12) Strong transversality condition. If f satisfies Axiom A then f satisfies the strong transversality condition iff $W^s(\psi)$ and $W^u(\psi)$ are transversal for all $\psi \in M$.

(2.13) Definition. $f \in \mathrm{Diff}^r(M)$ is Morse–Smale iff:

(1) $\Omega(f)$ is finite,

(2) f satisfies Axiom A and the strong transversality condition.

That $\Omega(f)$ is finite means that it consists of a finite number of period points.

The first structural stability theorem for diffeomorphisms was Peixoto's.

(2.14) Theorem (Peixoto [23]). The structurally stable diffeomorphisms are open and dense in $\mathrm{Diff}^r(S^1)$ and are the Morse–Smale diffeomorphisms.

Here the language has changed but the theorem remains the same. This theorem marked a revival in the subject. It motivated the following problems:

(1) Find open and dense or generic properties in $\mathrm{Diff}^r(M)$.

(2) Are the Morse–Smale diffeomorphisms open and dense and structurally stable on all manifolds?

In response to (1) there are the Kupka–Smale theorem [12, 36] and Pugh's general density theorem, as well as some conjectures (see [32], for example).

In response to (2) we have seen that the Morse–Smale diffeomorphisms are not open and dense by Smale's examples and the Lefschetz formula argument. This still left:

(2a) Are the Morse–Smale diffeomorphisms structurally stable?
(2b) Are the structurally stable diffeomorphisms open and dense?

In response to (2a) there are the theorems of Palis [19] and Palis–Smale [21], which we may state as:

(2.15) Theorem (Palis–Smale). The Morse–Smale diffeomorphisms are precisely the structurally stable diffeomorphisms with a finite Ω.

And because of this theorem and the Ω-stability theorem, the conjectures of Smale [40] and Palis–Smale [21]:

(2.16) Conjecture (Palis–Smale). $f \in \mathrm{Diff}^r(M)$ is structurally stable iff f is Axiom A and satisfies the strong transversality condition.

Robbin [25] proved half of this conjecture for f at least C^2, and Robinson [26] relaxed the hypothesis to C^1.

(2.17) Theorem (Robbin–Robinson). If f is Axiom A and satisfies the strong transversality condition, then f is structurally stable.

Once again one can prove:

(2.18) Theorem [40]. If f is structurally stable and satisfies Axiom A, then f satisfies the strong transversality condition.

So the real problem is: Does structural stability imply Axiom A?

Of course, if Ω-stability implies Axiom A so does structural stability. It is almost unthinkable that one would and the other would not.

As far as (2b) goes, Smale [38] showed in 1965 that structurally stable diffeomorphisms were not, in general, dense in $\mathrm{Diff}^r(M)$ with the C^r topology. He invented the notion of Ω-stability and, in 1968, Abraham and Smale [1] showed that Ω-stable diffeomorphisms are also not dense in the C^r topology. There are still a few concepts of stability that might prove dense, such as topological Ω-stability [11] or the topological structural stability of attractors of Thom. Both of these concepts include diffeomorphisms that are not Ω-stable. But it is certainly conceivable that a much weaker notion of stability would be required for density.

As most of the promising approaches to the problem of whether Ω-stability implies Axiom A rely on the periodic points, it seems reasonable to pose a seemingly more general problem.

Problem 1. Let $f \in \mathrm{Diff}^1(M)$. Suppose that there is a neighborhood U_f of f in $\mathrm{Diff}^1(M)$ such that $N_n(g) = N_n(f)$ for all $g \in U$ and all n. Does f satisfy Axiom A?

Ω-stable diffeomorphisms have this property. And the Axiom A diffeomorphisms that have this property are Ω-stable by Palis's result [20]. A diffeomorphism with this property might reasonably be called *zeta-function stable*. So we are asking: Is zeta-function stability $\Leftrightarrow$ Axiom A no cycles $\Leftrightarrow$ Ω-stability?

III. The Lefschetz Fixed-Point Formula and Stability

If f is an Axiom A diffeomorphism, Bowen [4] has proved that

$$h(f) = \limsup \,(1/n) \log N_n(f),$$

where $h(f)$ is the topological entropy of f. I will not define topological entropy here but will simply use $h(f)$ for $\limsup\,(1/n) \log N_n(f)$ when f satisfies Axiom A.

If $f\colon M \to M$ is continuous, we can define $s(f_*)$ to be the spectral radius of $f_*\colon H_*(M, R) \to H_*(M, R)$, that is, $s(f_*) = \max |\lambda|$, where the max is taken over all eigenvalues of f_{*i} and all i. It is not difficult to see that

(I) $\log s(f_*) = \limsup\,(1/n) \log |\sum \mathrm{tr}\, f_{*i}^n|$.
(II) Let $l(f) = \limsup\,(1/n) \log |\sum (-1)^i \mathrm{tr}\, f_{*i}^n|$.

$\log s(f_*) \geq l(f)$ because there is no alternation of signs in (I), whereas there is in (II).

Example 4. Let $f\colon M \to M$, and let $\theta\colon S^1 \to S^1$ be an irrational rotation. Let $g\colon M \times S^1 \to M \times S^1$ be $f \times \theta$. Thus by the Kunneth formula, $\mathrm{Tr}\, g_{*i}^n = \mathrm{tr}\, f_{*i}^n + \mathrm{tr}\, f_{*i-1}^n$. Because of the alternations of signs in (II), $\sum (-1)^i \mathrm{tr}\, g_{*i}^n = 0$ for all n and $l(g) = \log(0) = -\infty$. The eigenvalues of the

g_{*i} are the same as the eigenvalues of the f_{*i}, so $s(f_*) = s(g_*)$ and $\log s(f_*) = \log s(g_*)$. If $f\colon T^2 \to T^2$ were the Thom diffeomorphism then $g\colon T^3 \to T^3$, $l(g) = -\infty$, and $\log s(g_*) = \log [(3 + \sqrt{5})/2]$.

We can restate Proposition 2.4 as

(3.1) Proposition. If $g \in \mathrm{Diff}^r(M)$ is zeta-function stable, then

$$\limsup (1/n) \log N_n(g) \geq l(g).$$

But, in fact, one should be able to do much better:

(3.2) Conjecture. If $g \in \mathrm{Diff}^1(M)$ is zeta-function stable, then

$$\limsup (1/n) \log N_n(g) \geq \log s(g_*).$$

It is difficult to see how to attack this conjecture without the tools of Axiom A and no cycles, but perhaps some sort of algebraic approximation arguments would do the trick. Bob Williams and I have recently proved this conjecture for Axiom A and no-cycle diffeomorphism by proving the following, which was conjectured in [31] and [33].

(3.3) Theorem [35]. If $f \in \mathrm{Diff}^r(M)$ satisfies Axiom A and has no cycles, then

$$h(f) \geq \log s(f_*).$$

This theorem was previously known for Morse–Smale diffeomorphism [28, 33] and 0-dimensional Axiom A and no-cycle diffeomorphisms [5].

Let us give a corollary that includes the case of Morse–Smale diffeomorphisms.

(3.4) Corollary. Suppose $f \in \mathrm{Diff}^1(M)$ is zeta-function stable and has only finitely many periodic points; then every eigenvalue of $f_*\colon H_*(M, R) \to H_*(M, R)$ is a root of unity.

To see this corollary we note that each periodic point of f must be hyperbolic. By Pugh's theorem a nearby g will have its nonwandering set equal to the set of periodic points, so it will satisfy Axiom A. By Palis's argument g will have no cycles. By the theorem $h(g) \geq \log s(g_*) = \log s(f_*)$ and $h(g) = 0$. So every eigenvalue of f_* has modulus 1. As the f_{*i} are integral matrices, every eigenvalue is a root of unity.

If we return to Example 4 for a moment we see that $g\colon T^3 \to T^3$ has no periodic points at all and this corresponds precisely to $L(g^n) = 0$ for all n in the Lefschetz trace formula. But any Axiom A and no-cycle f that is even homologous to g must have infinitely many periodic points; in fact, $h(f) \geq \log\,[(3 + \sqrt{5})/2]$. So we naturally have the question: Is g isotopic to an Axiom A and no-cycle diffeomorphism? Smale answered this question in 1971.

(3.5) Theorem (Smale [43]). Any $f \in \mathrm{Diff}^r(M)$ is isotopic to an Axiom A and no-cycle diffeomorphism and hence an Ω-stable diffeomorphism.

This theorem was generalized in [30] with all the details carried out in [33] and [44].

(3.6) Theorem. Any $f \in \mathrm{Diff}^r(M)$ is isotopic to an Axiom A and strong transversality diffeomorphism and hence a structurally stable diffeomorphism. Moreover, the isotopy may be chosen to be C^0 small, so the structurally stable diffeomorphisms are dense in $\mathrm{Diff}^r(M)$ with the C^0 topology.

This theorem shows that there are no topological obstructions to structural stability and Theorem (3.3) shows that the periodic point structure of the Axiom A and strong transversality diffeomorphisms (which are the only known structurally stable diffeomorphisms) will be very rich. In many cases the growth rate of N_n will be exponential, whereas the Lefschetz formula predicts no periodic points at all. Naturally, then one is led to ask if $\log s(f_*)$ is the best lower bound one can find for $h(f)$ in terms of the homology theory of f, when f is Axiom A and no-cycles. This problem in terms of simplest diffeomorphisms is discussed at some length in [31] and [33]. Here I will restrict myself to the Morse–Smale case, because the distinction between finite and infinite periodic points is the sharpest, and the Morse–Smale diffeomorphisms are the simplest. Here there is a partial converse to Corollary (3.4).

(3.7) Theorem [33]. Let $\dim M \geq 6$ and $\pi_1(M) = 0$. If $f \in \mathrm{Diff}^r(M)$ and every eigenvalue of f_* is a root of unity [that is, $\log s(f_*) = 0$], then there is an integer $n > 0$ such that f^n is isotopic to a Morse–Smale diffeomorphism.

A precise condition for f itself to be isotopic to a Morse–Smale diffeomorphism is given in [33], but is interesting that n cannot always be chosen equal to 1 in the theorem. There is a further obstruction related to the structure of the ideals in the ring of integers of the cyclotomic extensions of the rationals. This obstruction is still not well understood (see [33]), but it is finite.

So we are getting closer to understanding how the homological information about f gives lower bounds for $h(f)$ for any structurally stable f, and these lower bounds are sharper than the Lefschetz fixed-point formula will give us. Lower bounds are all that is possible. Smale's examples can be adapted to show that if $\dim M \geq 2$, $f \in \mathrm{Diff}^r(M)$, and $k > 0$, then there is an Axiom A and strong transversality diffeomorphism g that is isotopic to f and $h(g) > k$.

Let me close with a small discussion of a possible generalization of Theorem (3.3) in terms of topological entropy, which would give information about the orbit structure of every smooth map in terms of homological information. Here $h(f)$ will stand for the topological entropy of any continuous function; for definitions and a further discussion see [31].

(3.8) Conjecture [31]. Let $f\colon M \to M$ be a smooth map (or diffeomorphism); then $h(f) \geq \log s(f_*)$.

Here there is some information known. Example 2 shows that the conjecture fails for continuous maps; in fact, it fails for some piecewise linear homeomorphisms of manifolds, so smoothness is essential. We do know, however, as a special case of work of Manning that:

(3.9) Theorem (Manning [17]). Let $f\colon M \to M$ be continuous. Then $h(f) \geq \log s(f_{*1})$, where $f_{*1}\colon H_{*1}(M, R) \to H_{*1}(M, R)$.

(3.10) Corollary (Manning [17]). Let $f\colon M \to M$ be a homeomorphism, and let $\dim M \leq 3$. Then $h(f) \geq \log s(f_*)$.

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