CHARLES PUGH AND MICHAEL SHUB*

Stable ergodicity and partial hyperbolicity

Dedicated to the memory of Ricardo Mañé.

1. Introduction

By the results of Anosov in 1967 volume preserving uniformly hyperbolic systems are ergodic and are open. Thus they exhibit robust statistical behavior. Averages are the same for almost all points, not only for the system in question but also for any small perturbation which preserves the same volume. If the perturbation only preserves a close by volume, then the averages of continuous functions are still close by. On the other hand, in 1954, Kolmogorov announced that there are no ergodic Hamiltonian systems in a neighborhood of a completely integrable one. Completely integrable systems have no hyperbolic behavior at all.

In this paper we will review the results of [Grayson, Pugh and Shub, 1994], [Pugh and Shub, 1996], [Pugh, Shub and Wilkinson, 1996], and [Brezin and Shub, 1995] which study the mixed situation in which the system is only partially hyperbolic.

Our themes are:

1) A little hyperbolicity goes a long way toward guaranteeing ergodic behavior.
2) Stably ergodic systems are considerably more general than one might have feared from Kolmogorov's theorem.
3) Some hyperbolicity may be necessary for stable ergodicity.

We consider C^2 diffeomorphisms f of closed manifolds M which preserve a fixed smooth volume on M. We say that f is stably ergodic if there is a neighborhood U of f in the C^2 volume preserving diffeomorphisms of M such that every $g \in U$ is ergodic.

* Partially supported by an NSF grant.

Each of our main themes is developed in a section below. Finally in section 5, we suggest some generalizations to dissipative systems.

2. Partial Hyperbolicity and Ergodicity

The Main Theorem of this section gives sufficient conditions for a diffeomorphism to be ergodic. We find stably ergodic diffeomorphisms by finding open sets of diffeomorphisms satisfying these conditions. The theorem may be interpreted to say that even for systems which are not uniformly hyperbolic, the same phenomenon which produces chaotic behavior i.e. some hyperbolicity may also guarantee ergodicity.

Main Theorem: [Pugh and Shub, 1996] Let $f : M \to M$ be partially hyperbolic and dynamically coherent. Suppose that the stable and unstable bundles have the accessibility property and that the invariant bundles are sufficiently Hölder. Then f is ergodic.

The accessibility property is a concept from control theory which we apply to the strong unstable and stable foliations of a partially hyperbolic diffeomorphism. We will soon explain these concepts. Partially hyperbolic diffeomorphisms which are dynamically coherent have some more properties which we will eventually come to. All three distributions E^s, E^c and E^u, the strong stable, center and strong unstable sub-bundles of the tangent bundle, of C^2 diffeomorphisms are Hölder. That they are sufficiently Hölder is expressed in terms of relationships of the Hölder exponents and ultimately in terms of the contraction and expansion constants of the various natural invariant bundles for the derivative. We leave these details to be consulted in [Pugh and Shub, 1996] and [Pugh, Shub and Wilkinson, 1996], but note that foliations with C^1 tangent bundles are sufficiently Hölder. Partially hyperbolic systems and the accessibility property were to our knowledge first considered in [Brin and Pesin, 1974].

We say that a C^r diffeomorphism $f : M \to M$ is *partially hyperbolic* iff $r \geq 1$ and there is a continuous Tf-invariant direct sum decomposition

$$TM = E^s \dotplus E^c \dotplus E^u$$

where E^s and E^u are non-trivial, some Finsler $\| \ \|$ on TM and some real constants $0 < a < b < c < 1 < d < e < g$ such that

$$a\|v\| < \|Tf(v)\| < b\|v\| \text{ for } v \in E^s - \{0\}$$
$$c\|v\| < \|Tf(v)\| < d\|v\| \text{ for } v \in E^c - \{0\}$$
$$e\|v\| < \|Tf(v)\| < g\|v\| \text{ for } v \in E^u - \{0\}.$$

Since $Tf : E^c \to E^c$ may have some contraction and expansion E^s and E^u are strong contracting and strong expanding Tf invariant subbundles. Tangent to E^s and E^u are the strong contracting and strong expanding f invariant foliations which we will denote by W^s and W^u.

Given continuous sub-bundles $F, H \subset TM$ and points $m_0, m_1 \in M$ we say that m_1 is *accessible* from m_0 iff there is a continuous piecewise C^1 path $\phi[0,1]$ joining

m_0 and m_1 i.e. ϕ is continuous, $\phi(0) = m_0$, $\phi(1) = m_1$ and there are a finite number of reals $0 = t_0 < t_1 < \cdots < t_j = 1$ such that $\phi|[t_i, t_{i+1}]$ is C^1 embedding, and is tangent either to F or H, $i = 0, \ldots, j-1$. We say the pair F, H has the *accessibility property* iff for any $m_0, m_1 \in M$ m_1, is accessible from m_0.

Only connected manifolds can have the accessibility property. We are assuming throughout that M is connected.

A *partially hyperbolic diffeomorphism* is said to have the *accessibility property* iff E^s, E^u has the accessibility property.

Accessibility is an equivalence relation on points in M. We say that the pair F, H has the *essential accessibility property* iff the only measurable subsets of M saturated by the equivalence relation have measure 0 or 1.

The *partially hyperbolic diffeomorphism* f is said to have the *essential accessibility property* iff E^s, E^u does. We could replace accessibility by the more general essential accessibility in the statement of Main Theorem, but accessibility is an easier property to verify.

Finally we define dynamically coherent. Let $f : M \to M$ be partially hyperbolic with splitting $E^s \oplus E^c \oplus E^u$. Then we say that f is *dynamically coherent* iff

1) there is a foliation $\mathfrak{L}$ (with C^1 leaves) tangent to E^c.
2) For any leaf L of $\mathfrak{L}$, $W^u(L)$ and $W^s(L)$ are unions of leaves of $\mathfrak{L}$.

We call $\mathfrak{L}$ the central foliation. The leaves of $\mathfrak{L}$ are C^1 and their tangent bundles vary continuously, in fact in a Hölder fashion. We call them central leaves, they are normally hyperbolic and hence have stable and unstable manifolds. A priori $\mathfrak{L}$ is not a C^1 foliation. For a C^1 foliation not only has C^1 leaves and a C^0 tangent bundle but it also has C^1 foliation charts.

We now give a condition which is simple to verify for a diffeomorphism to be partially hyperbolic and dynamically coherent.

Theorem 2.5. [P-S] Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism as above. If there is a C^1 foliation $\mathfrak{L}$ tangent to E^c, then f is partially hyperbolic and dynamically coherent, and so are all sufficiently small C^1 perturbations of f.

3. Genericity of Stable Ergodicity?

We do not know if Hölder smoothness without any further hypothesis is sufficient for the Main Theorem of the last section. For C^2 diffeomorphisms some Hölder smoothness follows from partial hyperbolicity. Nor do we know if dynamically coherent is essential. Partially hyperbolic may suffice. Also it is quite likely that the accessibility property is generic. So we conjecture

Conjecture 1: [Pugh and Shub, 1996] Among the partially hyperbolic C^2 volume preserving diffeomorphisms of M the stably ergodic are open and dense.

In particular, Conjecture 1 would imply that the generic C^2 volume preserving perturbation of an ergodic automorphism of the torus is ergodic (see [Grayson, Pugh and Shub, 1994] for some discussion of this). It also would imply that if $A : M \to M$ is a volume preserving C^2 Anosov diffeomorphism then the generic C^2 volume preserving perturbation of $A \times id : M \times N \to M \times N$ is stably ergodic for any compact manifold N. See [Bonatti and Diaz, 1994] for the rather striking topological transitivity version of this second case of Conjecture 1.

We do know a large class of examples of partially hyperbolic diffeomorphisms which are stably ergodic. The time one map of the geodesic flow on a compact surface of constant negative curvature is the most classically studied partially hyperbolic diffeomorphism and it has the accessibility property. In [Grayson, Pugh and Shub, 1994] we proved that it is stably ergodic. Amie Wilkinson [Wilkinson, 1995] removed the hypothesis that the negative curvature be constant. In n-dimensions we have:

Theorem 3. [Pugh and Shub, 1996] The time one map of the geodesic flow on the unit tangent bundle of a compact n-manifold of constant curvature $k, k < 0$ is stably ergodic - it is ergodic and so are all C^2 small volume preserving perturbations of it.

We have also a class of examples which come from the theory of homogeneous spaces of Lie groups. We will assume that our spaces are of the form G/B where G is a connected Lie group and B is a closed subgroup which, in addition, is *admissible* in a certain technical sense (see [Brezin and Shub, 1995]) which we will not make precise here. If G is nilpotent, solvable or semi-simple or if B is discrete then the admissibility condition is satisfied.

For $a \in G$ let L_a denote left translation by a i.e. $L_a(h) = ah$ for all $h \in G$. Then L_a induces a map on G/B which we call L_a as well. Given an automorphism A of G and $a \in G$ we call $L_a A : G \to G$ an affine diffeomorphism of G, we also denote this map by aA. If $A(B) = B$ then we continue to denote the induced map on G/B by $L_a A$ or aA and call it an affine diffeomorphism of G/B. We will assume that the Haar measure on G induces a finite measure on G/B which is invariant under left translation and that $A : G/B \to G/B$ is measure preserving.

Given an affine diffeomorphism $aA : G \to G$, aA induces an automorphism of the Lie Algebra $\mathfrak{g}$ of G by $ad(a)DA(e)$ where e is the identity of G. In particular, $ad(a)DA(e)$ is a linear map. Let $\mathfrak{g}^s$ and $\mathfrak{g}^u$ be the generalized eigenspaces of $\mathfrak{g}$ corresponding to the contracting and expanding eigenvalues of $ad(a)DA(e)$. Let $\mathcal{L} \subset \mathfrak{g}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^s$ and $\mathfrak{g}^u$. Then it is not hard to see [Pugh and Shub, 1996] that $\mathcal{L}$ is an ideal in $\mathfrak{g}$ which is $ad(a)DA(e)$ invariant. As an ideal $\mathcal{L}$ is tangent to the connected normal subgroup which we denote by H and call the hyperbolically generated subgroup of G.

Theorem 1: Let G/B be a compact manifold and aA be an affine diffeomorphism of G/B. Let r be a positive real. If the eigenvalues of $ad(a)DA(e)$ are sufficiently bunched near the three numbers 1, r or $\frac{1}{r}$ and $HB = G$, then aA is stably ergodic on G/B.

The theorem has examples for semi-simple groups. We specialize to SL(n,R).

Let Γ be a uniform discrete subgroup of $SL(n, R)$.

For $A \in SL(n, R)$ let $L_A : SL(n, R)/\Gamma \to SL(n, R)/\Gamma$ be given by left translation by A.

Theorem 2: [Pugh and Shub, 1996] The following four conditions are equivalent.

a) A has an eigenvalue with modulus different from 1.

b) L_A is partially hyperbolic and the stable and unstable bundles have the accessibility property.

c) The Lie Algebra generated by the contracting and expanding subspace of Adg is the whole Lie Algebra $SL(n, R)$.

d) L_A is stably ergodic among left translations of $SL(n, R)/\Gamma$.

4. Necessity

We say that aA is is stably ergodic under perturbations by left translations if there is a neighborhood U of a in G such that $a'A$ is ergodic for every $a' \in U$. We continue to assume that the pair G,B is admissible.

It is now easy to state our main theorem of this section.

Main Theorem: [Brezin and Shub, 1995] If an affine diffeomorphism is stably ergodic under perturbations by left translations then $\overline{HB} = G$ where H is the hyperbolically generated subgroup of G.

Remark (1). If $\overline{HB} = G$ then the affine diffeomorphism is ergodic, this is essentially Hopf's proof of the ergodicity of the geodesic flow. See [Pugh and Shub, 1996] where generalizations are proven in the C^2 category.

Remark (2). That $\overline{HB} = G$ is the same as the action of H on G/B being ergodic, which in this setting is the same as the essential accessibility property of [Pugh and Shub, 1996].

Remark (3). It is likely that stable ergodicity is actually equivalent to the condition that $\overline{HB} = G$. In the next two propositions we state some special cases of the theorem in which this is actually the case.

Proposition 1 [Brezin and Shub, 1995] Let G be a connected nilpotent Lie group and Γ a uniform discrete subgroup of G. Then the affine diffeomorphism aA of G/Γ is stably ergodic among left translations of G if and only if $\overline{H\Gamma} = G$.

Proposition 2 [Brezin and Shub, 1995] Let G be a connected semi-simple Lie group and Γ a lattice in G.

a) If G has no compact factors, then the affine diffeomorphism aA of G/Γ is stably ergodic among left translations of G if and only if $H = G$.

b) If G has compact factors then the affine diffeomorphism aA of G/Γ is stably ergodic among left translations of G if and only if $\overline{H\Gamma} = G$.

5. Dissipative Systems

Now we drop the condition that f be measure preserving and simply assume that f is C^r for some finite r. For $m \in M$, $W^s(m) = \{p \in M | d(f^n(m), f^n(p)) \to 0 \text{ as } n \to \infty\}$ and $W^u(m)$ is defined similarly. We define the partial order $>_1$, by $m >_1 n$ if $W^u(m)$ and $W^s(n)$ have non-empty intersection. Let $>$ denote the total order obtained by transitivizing $>_1$ and extending to a total order.

Conjecture 2: For the generic f and all $x \in M$ the set of $y \in M$ such that $x \geq y$ is a closed subset of M.

We say x is equivalent to y if $x \geq y$ and $y \geq x$.

Conjecture 3: Let dimension $M \geq 2$. For the generic finite dimensional submanifold V contained in $Diff^r(M)$ and almost every $f \in V$ the equivalence classes of points in the chain recurrent set of f are open in the chain recurrent set.

Conjecture 3 would give a finite spectral decomposition for f where each piece of the decomposition has something akin to the accessibility property.

REFERENCES

Bonatti, C. and Diaz, L. (1994), *Persistent Nonhyperbolic Transitive Diffeomorphisms*, preprint.

Brezin, J. and Shub, M. (1995), *Stably Ergodicity in Homogeneous Spaces*, preprint.

Brin, M.I. and Pesin, J.B. (1974), *Partially Hyperbolic Dynamical Systems (English Translation)*, Math. USSR Izvestia **8**, 177–218.

Grayson, M., Pugh, C. and Shub, M. (1994), *Stably Ergodic Diffeomorphisms*, Annals of Math **140**, 295-329.

Hopf, E. (1971), *Ergodic Theory and the Geodesic Flow on Surfaces of Constant Negative Curvature*, Bull. Amer. Mat. Soc. 77, 863–877.

Pugh, C. and Shub, M. (1996), *Stably Ergodic Dynamical Systems and Partial Hyperbolicity*, preprint.

Pugh, C., Shub, M. and Wilkinson, A. (1996), *Holder Foliations*, preprint.

Wilkinson, A. (1995), *Thesis - University of California, Berkeley*.

Charles Pugh
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720

Michael Shub
Mathematical Sciences Department
IBM Watson Research Center
Yorktown Heights, NY 10598