TN: **2546578**

Printed: 12/3/2020 2:19:31 PM

Journal Title: Dynamical Systems

Volume:
Issue:
Month/Year: 1973
Pages: 493-514

Article Author: Shub, M

Article Title: Stability and genericity for diffeomorphisms.

ODYSSEY REQUEST

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Stability and Genericity for Diffeomorphisms[†‡]

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Let M be a compact C^∞-manifold without boundary and $\mathrm{Diff}^r(M)$ the group of C^r diffeomorphisms of M with the C^r uniform topology. Generally speaking, the problem of dynamical systems for $\mathrm{Diff}^r(M)$ is to describe the global orbit behavior of "most" diffeomorphisms f. An excellent and important survey of this problem may be found in an article written by Smale in the summer of 1967 [34] and after in his survey lectures [33–36], although there have been some important theorems proven since then. We will try here to present a quick survey of some of these theorems with no pretense of being complete even for diffeomorphisms. Dynamical systems generally would deal with Lie group actions with or without conditions imposed, such as the one-parameter group actions arising from vector fields or Hamiltonian vector fields which are the origins of the subject. Much of what we say is taken from Smale's surveys.[§]

$\mathrm{Diff}^r(M)$ is a complete metrizable space, and by "most" we shall mean at least a Baire set in $\mathrm{Diff}^r(M)$, i.e., the countable intersection of

† AMS (MOS) 1970 Subject Classification: 58F99.

‡ This work was partially supported by NSF-GP 28375. It was done at IMPA, and I would like to thank the members for their hospitality.

§ It is a pleasure to acknowledge numerous helpful conversations with Smale and Newhouse.

open and dense sets, although open and dense would certainly be more satisfying. A Baire set of diffeomorphisms will be called generic, and a property P of diffeomorphisms which holds for all diffeomorphisms in a Baire set will also be called generic. Basically we are interested not only in describing a generic set but also in the changes in the orbit structure of diffeomorphisms g in a small neighborhood U of f in $\mathrm{Diff}^r(M)$, which are called perturbations of f, hence we will be led to various notions of "stability."

If $f \in \mathrm{Diff}^r(M)$ and $x \in M$, then $(f^n(x))_{n\in Z}$ is the orbit of x. f may have two types of orbits:

(1) $\{f^n(x)\}_{n\in Z}$ is finite, i.e., $\exists n \in Z - \{0\}$ such that $f^n(x) = x$, in which case we call x periodic;

(2) $\{f^n(x)\}_{n\in Z}$ is infinite, i.e., $- \exists n \in Z - \{0\}$ such that $f^n(x) = x$.

Up to the time of this conference the best general picture of the orbit structure of diffeomorphisms that we have is provided by the conclusions of Smale's spectral decomposition theorem. So proceeding from an ahistorical point of view we will present the conclusions of this theorem first.

Definition Let $f \in \mathrm{Diff}^r(M)$ and let $x \in M$. x is a nonwandering point of f, $x \in \Omega(f)$ or $x \in \Omega$, if for every open neighborhood U of x there exists an $n \in Z - \{0\}$ such that $f^n(U) \cap U \neq \varnothing$.

The periodic points of f are, of course, nonwandering, and in some sense the nonwandering points contain all the behavior at infinity. More precisely, if $x \in M$:

$$\alpha(x) = \{y \in M \mid \exists n_i \to -\infty \text{ such that } f^{n_i}(x) \to y\},$$

and

$$\omega(x) = \{y \in M \mid \exists m_i \to \infty \text{ such that } f^{m_i}(x) \to y\}.$$

The α and ω limits of any point $x \in M$, $\alpha(x)$ and $\omega(x)$, respectively, are also easily seen to be in Ω.

Definition We say that a diffeomorphism f has an Ω-decomposition if $\Omega(f)$ may be written as the finite disjoint union of closed invariant sets for f, $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$. Moreover, if $f \mid \Omega_i$ is topologically transitive for all i, i.e., $f \mid \Omega_i$ has a dense orbit for all i, then we say that f has a spectral decomposition.

If f has an Ω-decomposition then it is easily seen, as in Smale [34], that $\alpha(x) \subset \Omega_i$ for some fixed i, and $\omega(x) \subset \Omega_j$ for some fixed j.

Definition Let $f \in \mathrm{Diff}^r(M)$ have an Ω-decomposition. Then

$$W^s(\Omega_i) = \{x \in M \mid \omega(x) \subset \Omega_i\}$$

is called the stable set of Ω_i, and

$$W^u(\Omega_i) = \{x \in M \mid \alpha(x) \subset \Omega_i\}$$

is called the unstable set of Ω_i.

From the remarks above, we see:

Proposition Let $f \in \mathrm{Diff}^r(M)$ have an Ω-decomposition, $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$. Then M is the disjoint union of the $W^s(\Omega_i)$, $i = 1, \ldots, k$, and M is also the disjoint union of the $W^u(\Omega_i)$, $i = 1, \ldots, k$.

Thus we are beginning to have some picture of the orbit structure of diffeomorphisms f with Ω decompositions, $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$. Any wandering point x has its past history near some Ω_j, and its future history near some Ω_i, i.e., $x \in W^s(\Omega_i) \cap W^u(\Omega_j)$ for unique i and j; and if f has a spectral decomposition, each Ω_i is held together by a dense orbit.

Now given $f \in \mathrm{Diff}^r(M)$ with an Ω-decomposition, we may define a relation on the Ω_i as follows: $\Omega_i > \Omega_j$ if $(W^u(\Omega_i) - \Omega_i) \cap (W^s(\Omega_j) - \Omega_j) \neq \emptyset$, i.e., there is an x which comes from Ω_i and goes to Ω_j.

Definition Let $f \in \mathrm{Diff}^r(M)$ have an Ω-decomposition, $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$. We say that f has no cycles if $\Omega_{i_0} > \Omega_{i_1} > \cdots > \Omega_{i_j} = \Omega_{i_0}$ is impossible for any $j \geq 1$.

If f has an Ω-decomposition and no cycles we may reorder the Ω_i by defining $\Omega_i > \Omega_j$ iff there exists a sequence $\Omega_i > \Omega_{k_1} > \cdots > \Omega_{k_l} > \Omega_j$ and finally we may reindex the Ω_i such that $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ and $i < j \Rightarrow \Omega_i \ngtr \Omega_j$. Henceforth, we will assume that the Ω_i are indexed as above for any f with an Ω-decomposition and the no cycle property. Following Newhouse [15], we call this a filtration ordering of the Ω_i.

A diffeomorphism f with an Ω-decomposition and the no-cycle property has a somewhat understandable orbit picture as shown in Figure 1. The arrows indicate directions in which points are proceeding under positive iterations of f.

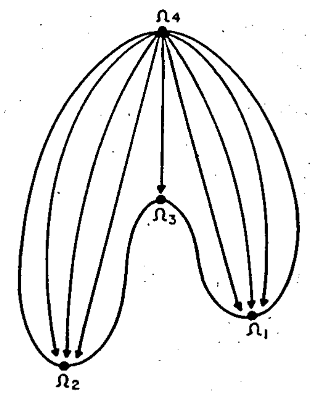

Figure 1

We remark that this picture has very little to do with the Ω_i themselves. It would be equally as valid for any k-disjoint closed invariant sets $\Lambda_1 \cdots \Lambda_k$ whose union contains Ω, or as Newhouse pointed out to the author, such that the α and ω limits of any point x lie in the union of the Λ_i. M would still be the disjoint union of the $W^s(\Lambda_i)$s and $W^u(\Lambda_i)$s, and the definition of no-cycles would still make sense, etc.

We have concentrated on Ω because Ω has been one of the main objects of study in dynamical systems, and we have singled out the topological transitivity because it seems important to us in terms of giving the Ω_i some unifying property. Of course, one might be interested in properties between Ω-decomposition and spectral decomposition; for example, one might wish to assume that Ω_i is indecomposable, i.e., it is not the union of two closed disjoint invariant sets.

Definition Let $f \in \mathrm{Diff}^r(M)$ a filtration for f is a sequence of compact manifolds with boundary $M = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$, such that $f(M_k) \subset$ Interior M_k. Given a filtration $K_i = \bigcap_{n \in Z} f^n(M_i$ $-$ Interior $M_{i-1})$ is the maximal invariant set contained in $M_i - M_{i-1}$. If $K_i = \Omega \cap (M_i - M_{i-1})$, for all i, we say that the filtration is a fine filtration for f. Finally, if we are given closed invariant disjoint sets $\Lambda_1, \ldots, \Lambda_k$, we say $M = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ is a filtration for $\Lambda_1, \ldots, \Lambda_k$ if $\Lambda_i = K_i$.

Proposition Let $M = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ be a filtration for $f \in \mathrm{Diff}^r(M)$. Then:

a. $\Omega \subset \bigcup_{i=1}^k K_i$;
b. M is the disjoint union of the $W^s(K_i)$ and $W^u(K_i)$;
c. there are no cycles and $i > j \Rightarrow K_j \not> K_i$;
d. $M_i \subset \bigcup_{j \le i} W^s(K_j)$;
e. given arbitrarily small neighborhoods U_i of K_i, $\exists m_i$ and $n_i > 0$ such that $f^{m_i} M_i - f^{-n_i}$ (Interior $M_{i-1}) \subset U_i$.

Proof (a) If $x \notin \bigcup_{i=1}^k K_i$, then $x \in M_j -$ Interior M_{j-1}, for some j, and there exists an n such that $f^n(x) \notin M_j -$ Interior M_{j-1}. So either $f^n(x) \in$ Interior M_{j-1}, in which case x is wandering, or $f^n(x) \in$ complement of M_j, which is invariant by f^{-1}, so x is wandering once again.
(b) follows from (a) and the above remarks.
(c) Since $f(M_i) \subset$ Interior M_i, K_i is in the interior of $M_i -$ Interior M_{i-1}, so $W^u(K_i)$ is contained in M_i, so if $W^u(K_j) \cap W^s(K_l) \neq \emptyset$, $j > l$.
(d) If $x \in M_i$, then $\omega(x) \subset \bigcup_{j \le i} K_j$, so that $M_i \subset \bigcup_{j \le i} W^s(K_j)$.
(e) As $\bigcap_{n \in Z} f^n(M_i -$ Interior$(M_{i-1})) = K_i$, $\exists n_1, n_2 > 0$ such that

$$\begin{aligned} U_i &\supset \bigcap_{-n_1 \le m \le n_2} (f^m(M_i - \text{Interior}(M_{i-1})) \\ &= \bigcap_{-n_1 \le m \le n_2} (f^m(M_i) - f^m(\text{Interior } M_{i-1})) \\ &\supset f^{n_2}(M_1) - f^{-n_1}(\text{Interior } M_{i-1}). \end{aligned}$$

As an immediate corollary, proven as by Pugh and Shub [24], we have

Corollary Let $f \in \mathrm{Diff}^r(M)$. Let $\Lambda_1, \ldots, \Lambda_k$ have a filtration. Then given neighborhoods U_i of Λ_i in M, there exists a neighborhood U of f in $\mathrm{Diff}^r(M)$ such that $g \in U \Rightarrow \Omega(g) \subset \bigcup_{i=1}^k U_i$. In particular, if f has a fine filtration, then for any neighborhood U_1 of $\Omega(f)$ in M there exists a neighborhood U_2 of f in $\mathrm{Diff}^r(M)$ such that $g \in U_2 \Rightarrow \Omega(g) \subset U_1$.

This corollary is valid for a C^0-neighborhood in the homeomorphisms of M and for fine filtrations says that f has no Ω-explosions, which is a weak sort of stability.

As a converse to the proposition we have:

Proposition Let Λ_i, $i = 1, \ldots, k$, be closed disjoint invariant sets for $f \in \mathrm{Diff}^r(M)$. Let

$$\alpha(x) \cup \omega(x) \subset \bigcup_{i=1}^k \Lambda_i, \qquad \forall x \in M,$$

and let the Λ_i have no cycle. Let $\Lambda_1, \ldots, \Lambda_k$ be a filtration ordering of the Λ_i. Then $\Lambda_1, \ldots, \Lambda_k$ has a filtration.

The proof of this proposition is a combination of arguments in Smale [34], Palis [17], and Pugh and Shub [24]. We will present it in [29]. If we specialize these theorems to Ω we get

Theorem Let $f \in \mathrm{Diff}^r(M)$. Then f has a fine filtration if and only if f has an Ω decomposition with no cycles.

As a corollary first proven in [24] we have

Corollary Let $f \in \mathrm{Diff}^r(M)$ have an Ω-decomposition, $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ with no cycles. Then given open sets U_i of Ω_i there exists a neighborhood U of f in $\mathrm{Diff}^r(M)$ such that $g \in U \Rightarrow \Omega(g) \subset \bigcup_{i=1}^k U_i$.

Filtrations for sets $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$, where Λ was not assumed to be Ω, were considered by Newhouse [15]. The above theorem was proven before the word fine filtration was invented by Smale at this conference. The idea was that the fine filtration or Ω-decomposition and no-cycle diffeomorphisms and especially the spectral decomposition and no-cycle diffeomorphisms were good objects to study. For example, one should be able to determine the Cech cohomology of the Ω_i from the filtration data, and the work of Conley and Easton [5] and Smale [32] is relevant here. Since filtrations are stable, i.e., a filtration for f is a filtration for all nearby g, one should have the same sort of information for the K_i for g. At the start of the conference the author conjectured that spectral decomposition and no-cycle diffeomorphisms were generic in $\mathrm{Diff}^r(M)$, even though we said this conjecture might be ridiculous. Since then, Newhouse has produced a counterexample to this conjecture in $\mathrm{Diff}^r(S^2)$, $r \geq 2$. Smale came up with the notion of a fine sequence of filtrations, (see [29], which still has many of the properties of a fine filtration and seems to include Newhouses example, so it stands a chance of being a generic property. Finally, let us say that many of these ideas are the outgrowth of conversations with Smale in Rio and Salvador. Newhouses paper [15] provides added background for the above theorems and he kindly checked through the proofs. The diffeomorphisms with the spectral decomposition and no-cycle properties by themselves describe a large class of diffeomorphisms and are still very interesting to study, so in the tradition of these survey talks we will still make the following, perhaps ridiculous, conjecture.

Definition Let $F \subset \mathrm{Diff}^r(M)$ be the diffeomorphisms with a fine filtration and let $S \subset \mathrm{Diff}^r(M)$ be the spectral decomposition and no-cycle properties diffeomorphisms.

Conjecture S is dense in F.

The spectral decomposition property was first considered as a consequence of Axiom A by Smale in [34], and in [35] he used Axiom A and the no-cycle property to prove the Ω-stability theorem (see below).

So far we have deemphasized stability properties. Historically this is somewhat backward, and we do not mean to neglect them. As late as Smale's survey article in 1967, he posed the problem of dynamical systems as follows: Let r be an equivalence relation on $\mathrm{Diff}^r(M)$ which is reasonable as far as orbit structure is concerned. Define a diffeomorphism f to be r stable if its r equivalence class contains a neighborhood of f. The problem then was to find an r relation whose stable elements are open and dense. This formulation was motivated by a previous statement of the problem which was to find a generic or open and dense set U of elements in $\mathrm{Diff}^r(M)$ such that the orbit structure of U could somehow be described qualitatively by discrete numerical and algebraic invariants. Since $\mathrm{Diff}^r(M)$ is separable, the later formulation would give hope of finding the invariants, since an open and dense set of diffeomorphisms would have a countable number of r equivalence classes. Since then Smale has restated the problem [36] as each equivalence relation has failed to have an open and dense set of stable elements. Of course, the stronger the equivalence relation the more well behaved the stable elements under perturbation and in some sense the more "physical" the system described, because one cannot be sure of one's initial data. The first such equivalence relation comes from Andronov–Pontryagin; it is an isomorphism of the orbit structure.

Definition Let $f \in \mathrm{Diff}^r(M)$, $g \in \mathrm{Diff}^r(N)$; then f and g are topologically conjugate if there exists a homeomorphism $h: M \to N$ such that $hf = gh$. The elements in $\mathrm{Diff}^r(M)$ which are stable for the topological conjugacy equivalence relation are called structurally stable.

When Smale showed that structurally stable diffeomorphisms were not dense, he introduced a weaker equivalence relation which is an isomorphism of the orbit structures on Ω, "where the action is."

Definition Let $f \in \mathrm{Diff}^r(M)$, $g \in \mathrm{Diff}^r(N)$. Then f and g are Ω conjugate if there exists a homeomorphism $h: \Omega(f) \to \Omega(g)$ such that $hf = gh$. The elements in $\mathrm{Diff}^r(M)$ which are stable for the Ω-conjugacy equivalence relation are called Ω-stable.

In the Spring of 1968, Abraham and Smale [1], showed that the Ω-stable diffeomorphisms are not open and dense.

We will delay giving any examples, especially as they are readily found in the literature (see Smale [34] and Palis [19]), and will instead give an abstract presentation of some of these systems. Since structural stability implies Ω-stability, we will begin with Smale's Ω-stability theorem.

DEFINITION Let $f \in \mathrm{Diff}^r(M)$ and let Λ be a closed invariant set for f. Then Λ is hyperbolic if $TM \mid \Lambda = E^s \oplus E^u$, $T_\Lambda f\colon E^s \to E^s$, and $T_\Lambda f\colon E^u \to E^u$, E^s and E^u are continuous subbundles of TM, and there exist constants $C > 0$, $\lambda < 1$, such that

$$\| T_\Lambda f^n \mid E^s \| \leq C\lambda^n \quad \text{and} \quad \| T_\Lambda f^{-n} \mid E^u \| \leq C\lambda^n \quad \text{for all} \quad n > 0.$$

So, E^s is contracted by $T_\Lambda f$ and E^u is expanded by $T_\Lambda f$. If Λ is a periodic orbit of period p, then Λ is hyperbolic if and only if $T_x f^p$ has no eigenvalue of absolute value one for $x \in \Lambda$, in which case x is called a hyperbolic periodic point.

DEFINITION (Smale) $f \in \mathrm{Diff}^r(M)$ satisfies Axiom A if and only if

a. Ω is hyperbolic;

b. The periodic points of f are dense in Ω, $\Omega(f) = \overline{\mathrm{Per}\,f}$.

SPECTRAL DECOMPOSITION THEOREM (Smale [34]) If f satisfies Axiom A, f has a spectral decomposition.

Ω-STABILITY THEOREM (Smale [34, 35]) If f satisfies Axiom A and has no cycles, then f is Ω-stable.

The proof of this theorem proceeded via the filtration theorem and a local analysis of hyperbolic Ωs. Actually the theorem Smale proved was somewhat stronger because of his no-cycle condition. Smale stated the following and it follows from the work of Hirsch *et al.* [10].

PROPOSITION If f satisfies Axiom A, and $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ is the spectral decomposition, then $W^s(\Omega_i) \cap W^u(\Omega_i) = \Omega_i$ for all i.

This proposition says that an Axiom A f has no 1-cycle.

Smale has conjectured:

CONJECTURE (Smale) $f \in \mathrm{Diff}^r(M)$ is Ω-stable if and only if f satisfies Axiom A and has the no-cycle property.

To this end Palis has proven

THEOREM (Palis [18]) If f satisfies Axiom A and if f is Ω stable, then f has the no-cycle property.

Recently Franks has defined a notion of differentiable Ω-stability. Let $H(\Omega, M)$ be the set of homeomorphisms of Ω into M with the C^0 topology and metric ϱ. Let $d(f, g)$ denote the C^0-distance between two elements of $\mathrm{Diff}^r(M)$.

DEFINITION Let $f \in \mathrm{Diff}^r(M)$. f is differentiably Ω-stable if there exists a function $\phi\colon U_f \to H(\Omega, M)$, where U is a neighborhood of f, ϕ is differentiable at f, $\phi(f) =$ the inclusion of Ω, and $\phi(g)$ is an Ω-conjugacy between f and g for any $g \in U$.

THEOREM (Franks [7]) Let $f \in \mathrm{Diff}^1(M)$. Then f satisfies Axiom A and has the no-cycle property if and only if f is differentiably Ω-stable and $\exists K > 0$ such that $\varrho(\phi(g), \mathrm{incl}_\Omega) \leq K\, d(f, g)$.

While this important theorem certainly classifies the Axiom A no-cycle property diffeomorphisms in $\mathrm{Diff}^1(M)$ in terms of an Ω stability property, Smale's conjecture still stands and is still a very interesting problem. Putting some conditions on the conjugacy is reasonable, and previously for structural stability one used to ask for a conjugacy function such as ϕ but which was just continuous at f. Another interesting approach to the conjecture may be found in Franks [6], but we will not go into it here.

Turning our attention to structurally stable diffeomorphisms, we begin by defining the stable set of a point as in Smale [33]. We say that for $x, y \in M$ and a fixed diffeomorphism f, $x \sim_s y$ iff $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$, and $x \sim_u y$ iff $d(f^n(x), f^n(y)) \to 0$ as $n \to -\infty$. We denote by $W^s(x)$ and $W^u(x)$ the equivalence classes of x for these relations. Note that the W^ss and W^us partition the manifold M.

THEOREM ([34, 10]) If f satisfies Axiom A then the W^ss and W^us are smooth 1–1 immersed disks.

Smale has raised the following very interesting question.

QUESTION Are the $W^s_{(x)}$s generically smooth manifolds $\forall x \in M$?

In the case of a periodic point x of period p, $W^s(x) = \{y \in M \mid f^{np}(y) \to x \text{ as } n \to \infty\}$ and we have

Stable Manifold Theorem If x is a hyperbolic periodic point of $f \in \mathrm{Diff}^r(M)$, $W^s(x)$, $(W^u(x))$ is a C^r 1–1 immersed cell of dimensions $s(u)$ which is tangent to E^s (E^u) at x.

Much of the analysis that goes into the proof of the Ω-stability theorem and the theorem immediately above is concerned with a generalization of this theorem to hyperbolic sets (see Hirsch and Pugh [8] and Hirsch *et al.* [10]).

Definition Let f be Axiom A, then f satisfies the strong transversality condition if and only if $W^s(x)$ and $W^u(x)$ are transversal for all $x \in M$.

If $\Omega(f)$ is finite, then it consists of a finite number of periodic points. If it satisfies Axiom A, then each of these periodic points is hyperbolic. The spectral decomposition is then just each periodic orbit. The stable set of the orbit (as defined after the Ω decomposition property) is the union of the stable manifolds of the points in the orbit. The strong transversality condition then becomes $W^s(p)$ only intersects $W^u(q)$ transversally for any two periodic points p and q of f. Thus we have the usual definition of Morse–Smale diffeomorphisms, so

Definition $f \in \mathrm{Diff}^r(M)$ is Morse–Smale if and only if $\Omega(f)$ is finite and f satisfies Axiom A and the strong transversality property.

We remark that it can be seen that a diffeomorphism which satisfies Axiom A must satisfy the strong transversality property in order to be structurally stable. Also the strong transversality property implies the no-cycle property, and Smale first used a similar property in proving the Ω-stability theorem.

There is one case in which the problem of classification of an open and dense set of diffeomorphisms has been solved. That is the beautiful theorem of Peixoto which raised so many hopes and questions.

Theorem (Peixoto [21]) The Morse–Smale diffeomorphisms are open and dense in $\mathrm{Diff}^r(S^1)$ and are the structurally stable diffeomorphisms.

This theorem gives the existence of a countable number of models which describe the topological conjugacy classes of an open and dense set of diffeomorphisms of S^1.

Morse–Smale diffeomorphisms are the simplest diffeomorphisms to understand on any manifold. Their existence is assured by the following

constructions: Fix a Riemannian metric on M and let $f: M \to R$ be a differentiable function. Let $-\mathrm{grad}(f)$ be minus the gradient vector field of f, and let φ_t be its associated one-parameter group of diffeomorphisms. φ_t has the property that $f(\varphi_t(x))$ is a strictly decreasing function of t unless x is a critical point of f, in which case $\varphi_t(x) = x$, $\forall t$ (see Figure 2).

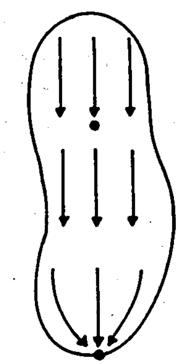

Figure 2

Now fix $t = 1$, and consider the diffeomorphism $\varphi_1: M \to M$. From the above, it follows that $\Omega(\varphi_t) =$ the singularities of $f =$ the fixed points of φ_1. Now by a theorem of Morse any $f: M \to R$ may be approximated by one with nondegenerate critical points. A nondegenerate critical point is defined as a critical point where $D^2 f$ defines a nondegenerate quadratic form on $T_x M$. A nondegenerate critical point is isolated and is a hyperbolic fixed point of φ_1. So if $f: M \to R$ has only nondegenerate critical points, we have $\Omega(\varphi_1)$ is finite and hyperbolic and, in fact, consists of the fixed points of φ_1. Now it follows from a theorem of Smale that an $f: M \to R$ which has only nondegenerate critical points may be approximated by one such that φ_1 has the strong transversality property.

The usual height function on the torus T^2, for example, has four hyperbolic fixed points for φ_1 but fails to satisfy the strong transversality condition because $W^u(x_3) - x_3 = W^s(x_2) - x_2$ and they are both one-cells (see Figure 3). By tilting the torus a little we may make them disjoint. Since the stable and unstable manifolds of periodic points are taken into stable and unstable manifolds of periodic points by a topological conjugacy, we see that the usual height function gives a φ_1 that is not structurally stable because of the lack of transversality. It is easy to see, however, that the φ_1 obtained by tilting the torus a little is structurally stable. Note that the original φ_1 is Axiom A and has no cycles, so it is Ω-stable.

We have dwelt this long on this example of a Morse–Smale φ_1, although we have not generally been presenting the rich multitude of examples, because these φ_1 seem to have motivated the spectral decomposition picture. What happens is that a fixed point is replaced by a more complicated Ω_i. We keep the hyperbolicity and transversality with Axiom A and strong transversality, drop the transversality for Axiom A and strong transversality, drop the transversality for Axiom A and no cycles, and finally drop the hyperbolicity for fine filtrations or the spectral decomposition and no-cycle property.

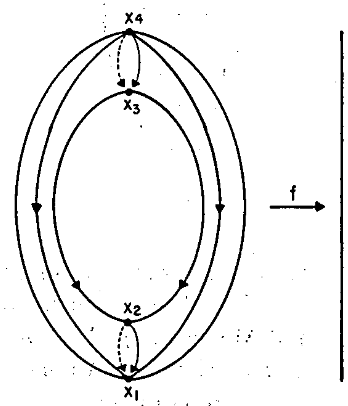

Figure 3

We will try to make this more precise as follows, but for the best treatment we will have to speak about flows. Recall that a C^r-vector field X on M is a C^r-section of the tangent bundle TM. X gives rise to a C^r 1-parameter group of diffeomorphisms $X_t: M \to M$, $t \in R$, with the properties that

$$X_t \circ X_s = X_{t+s} \quad \text{and} \quad \frac{d}{dt} X_t(x)\,|_{t=0} = X(x).$$

A subset $\Lambda \subset M$ is invariant for X_t if and only if $X_t(\Lambda) = \Lambda$, $\forall t \in R$.

Definition A filtration for a 1-parameter group of diffeomorphisms $\varphi_t: M \to M$ is a sequence of compact submanifolds with smooth boundary $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that $\varphi_t(M_i) \subset \text{Int}(M_i)\ \forall t > 0$,

and $\varphi_t(x)$ is transversal to ∂M_i, $\forall x \in M$ and $0 < i < k$. A filtration for φ_t is called fine if

$$K_i = \bigcap_{t>0} \varphi_t(M_i - M_{i-1}) = \Omega \cap (M_i - M_{i-1}).$$

What we have said about filtrations for diffeomorphisms applies equally well for one-parameter groups, where if $x \in M$,

$$\omega(x) = \{y \in M \mid \exists t_i \to \infty, t_i \in R \text{ and } \varphi_{t_i}(x) \to y\},$$

etc.

Definition Let X be a C^r vector field on M, and let Λ be a closed invariant set for X_t. $L_X: M \to R_+$ is a Lyapunov function for (X, Λ) if the critical set of L_X equals Λ, $\langle \text{grad } L_X, X\rangle < 0$ on $M - \Lambda$ for some Riemannian metric on M, and L_X is differentiable of class at least dim M. If $\Lambda = \Omega$ we simply say L_X is a Lyapunov function for X.

The following theorem was first indicated in [24] and its proof using [41] is essentially given there.

Theorem Let X be a C^r vector field on M, and let $M = M_k \supset \cdots \supset M_0 = \emptyset$ be a filtration for X_t. Then there exists a Lyapunov function L_X for (X, K), where $K = \bigcup_{i=1}^k K_i$ and $L_X(K_i) = i$.

As a corollary we have:

Corollary X_t has a fine filtration if and only if X has a Lyapunov function with a finite number of critical values.

In this case the Lyapunov function by [41] can be made C^∞. In general we suppose a vector field will have a Lyapunov function iff it has a fine sequence of filtrations. In particular, if X_t is Axiom A′ (see Smale [34] for definitions) we have the theorem stated by Pugh and Shub [24].

Theorem ([24]) If X_t is Axiom A′ and has no cycles, then X has a C^∞ Lyapunov function with a finite number of critical values.

The study of Lyapunov functions for vector fields, especially their bifurcations seems very important. It would be interesting to know if Axiom A′ and no-cycle vector fields have special Lyapunov functions. The energy functions of Meyer [13] are special Lyapunov functions for Morse–Smale vector fields. The following is an important and open problem:

CONJECTURE The vector fields which have Lyapunov functions are generic.

Now if we think of a diffeomorphism as a section of its suspension we get:

DEFINITION Let $f \in \mathrm{Diff}^r(M)$ and let Λ be a closed invariant set for f. $L_f\colon M \to R_+$ is a Lyapunov function for (f, Λ) if the critical set of $L_f \supset \Lambda$, $L_f(f(x)) \leq L_f(x)$, equality holds if and only if $x \in \Lambda$, and L_f is differentiable of class at least $\dim M + 1$. If $\Lambda = \Omega$ we simply say L_f is a Lyapunov function for f.

As a consequence we have the following:

THEOREM Let $f \in \mathrm{Diff}^r(M)$ and let $M = M_k \supset \cdots \supset M_1 \supset M_0 = \emptyset$ be a filtration for f, then there exists a C^∞ Lyapunov function L_f for (f, K), where $K = \bigcup_{i=1}^k K_i$ and $L_f(K_i) = i$.

CONJECTURE The diffeomorphisms which have Lyapunov functions are generic.

Mañé independently is studying similar problems departing from Auslander's R-set (see [3, 4]). He has a program which could prove the above conjecture, and talking to him was very interesting. We could have constructed the Lyapunov function first for an Ω-decomposition with no-cycles and then obtained the filtration from L_f by taking regular values, but the proof of the existence of L_f almost gives the filtration. Once again we suppose that the diffeomorphisms which have Lyapunov functions are the same as those with a fine sequence of filtrations. It would be interesting to know when, if not always, we could assume that L_f is critical exactly on Λ in the definition of a Lyapunov function for (f, Λ).

Smale first defined Morse–Smale diffeomorphisms in his work [30] in which he proved the no-cycle property and more from the strong transversality condition. Rosenberg later first isolated the no-cycle condition [27] in proving a generalization of the Morse inequalities.

Let $f \in \mathrm{Diff}^r(M)$ be Morse–Smale. Let C_i be the number of periodic points x of f with dimension $W^u(x) = i$. Let B_i be the ith Betti number of M.

THEOREM (Smale [30]) Let $f \in \mathrm{Diff}^r(M)$ be Morse–Smale. Then

$$C_0 \geq B_0,$$
$$C_1 - C_0 \geq B_1 - B_0,$$
$$\vdots$$
$$\sum_{i=0}^{\dim M} (-1)^i C_i = \sum_{i=0}^{M} (-1)^i B_i = \chi(M),$$

the Euler characteristic of M.

This theorem has previously been one of the few theorems relating the actual orbit structure of a diffeomorphism to the topology of the manifold. The finer analysis of the relations of the C_i to the B_i has somehow more traditionally been considered differential topology for the case of gradient fields at least. Of course, Smale's H-cobordism theorem is the outstanding example of this. One might ask for Morse type inequalities for more general Ωs, or even for Ωs with a given filtration.

Returning to Stability properties, Palis proved:

THEOREM (Palis [17]) The Morse–Smale diffeomorphisms are open in $\mathrm{Diff}^r(M)$.

Then Palis and Smale proved:

THEOREM (Palis and Smale [20]) Let $f \in \mathrm{Diff}^r(M)$ be Morse–Smale. Then f is structurally stable.

They conjectured in this paper:

CONJECTURE (Palis and Smale [20]) $f \in \mathrm{Diff}^r(M)$ is structurally stable if and only if f is Axiom A and satisfies the strong transversality condition.

In the case that Ω is finite, the theorems of Palis [18], Palis and Smale [20], and Smale [35], give a complete answer to the stability conjectures.

THEOREM Suppose $f \in \mathrm{Diff}^r(M)$ and $\Omega(f)$ is finite. Then:

a. f is Ω-stable if and only if f is Axiom A and has the no-cycle property;
b. f is structurally stable if and only if f is Axiom A and satisfies the strong transversality condition.

Last year Robbin proved:

THEOREM (Robbin [26]) Let $f \in \mathrm{Diff}^2(M)$ be Axiom A and satisfy the strong transversality condition. Then f is structurally stable in $\mathrm{Diff}^1(M)$.

Robbin's technique is to find a fixed point in a Banach space of functions. Recently, Melo using techniques closer to Palis and Smales original proof has improved the theorem for 2-manifolds.

THEOREM (Melo [12]) Let $f: M^2 \to M^2$ be Axiom A and satisfy the strong transversality condition, where $f \in \mathrm{Diff}^1(M^2)$. Then f is structurally stable.

Starting from the Ω-stability theorem one can see that the Axiom A and strong transversality diffeomorphisms are open. But we have said nothing about examples. Recently, Smale has proven:

THEOREM (Smale [37]) Every $f \in \mathrm{Diff}^r(M)$ is isotopic to a C^∞ Axiom A and strong transversality diffeomorphism, and hence to a structurally stable diffeomorphism.

As the proof of this theorem is contained in this volume we will make no remarks about it except to note that it gives loads of examples of Axiom A and strong transversality diffeomorphisms which are not Morse–Smale. In fact, in the light of this theorem Smale has made a conjecture which was first raised as a problem by him in [34].

We recall that a Morse–Smale diffeomorphism is called gradient-like if $W^s(p) \cap W^u(q) \neq \emptyset$ implies that $\dim W^u(q) > \dim W^u(p)$, for any two periodic points p and q of f.

CONJECTURE (Smale) f is homotopic (isotopic) to a gradient-like Morse–Smale diffeomorphisms if and only if f^n is homotopic (isotopic) to the identity of M for some n. Smale has told the author that if f is homotopic to a gradient-like Morse–Smale diffeomorphism, then f^n is homotopic to the identity for some n.

The problem of finding necessary and sufficient conditions for a diffeomorphism to be homotopic (isotopic) to a Morse–Smale diffeomorphism seems more difficult. In these proceedings we have proven [28]

THEOREM If f is homotopic to a Morse–Smale diffeomorphism then $f_*: H_*(M, R) \circlearrowleft$ is quasi unipotent.

The converse of this theorem should be true at least in the case where M is the n-torus T^n.

In this Bourbaki seminar talk, Smale restated the problem of dynamical systems more or less as follows:

Find an increasing sequence of open sets $U_1 \subset U_2 \subset U_3 \subset \cdots \subset U_k \subset \mathrm{Diff}^r(M)$ such that k is not too large, the f_i in U_i have decreasing regularity properties with strong regularity properties for small i, and U_k is generic. He proposed given the state of the subject at that time that:

$U_1 = \{f \in \mathrm{Diff}^r(M) \mid \Omega(f)$ is finite and f satisfies Axiom A and the strong transversality condition$\}$,

$U_2 = \{f \in \mathrm{Diff}^r(M) \mid f$ satisfies Axiom A and the strong transversality condition$\}$,

$U_3 = \{f \in \mathrm{Diff}^r(M) \mid f$ satisfies Axiom A and has the no-cycle property$\}$,

$U_4 = \{f \in \mathrm{Diff}^r(M) \mid f$ has all the known generic properties$\}$.

The reader will easily be able to translate the conjectures about structural and Ω stability in terms of elements of U_1, U_2, and U_3, as in Smale [36]. Somehow these inclusions of the U_i seem like a filtration of $\mathrm{Diff}^r(M)$. On the other hand, it seems important somehow to uniformize Smale's isotopy at least for almost all diffeomorphisms. Smale's proof proceeds by picking a handle decomposition of the manifold. Bob Williams and the author worked out that if one picks the "wrong" handle decomposition of T^2, for example, his process can isotop a Morse–Smale diffeomorphism into an Axiom A and no-cycle diffeomorphism with an infinite Ω, which seems like it is complicating the problem. Thus one would somehow like to vary the handle decomposition with the diffeomorphism to give the most natural picture. Although this is not a precise statement of anything, it certainly seems relavant to the problem of which diffeomorphisms are isotopic to Morse–Smale diffeomorphisms and Smale's conjecture.

Conversations with Smale and Palis have been useful in formulating the following precise

CONJECTURE There exists a one-parameter group φ_t on $\mathrm{Diff}^r(M)$ with the following properties:

1. φ_t has only isolated critical points;
2. there exists an open and dense set $U \subset \mathrm{Diff}^r(M)$ such that given $g \in U$, there exists a $t_0 \in R$ with $\varphi_t(g)$ structurally stable for all $t > t_0$.

Relating this conjecture to the U_i of Smale, we think of it as follows: Is there a positive real-valued function $H\colon \mathrm{Diff}^r(M) \to R$ such that:

1. H has isolated critical points;
2. $\exists C_i > 0$ such that U_i is open and dense in $H^{-1}[0, C_i]$, for $i \leq 3$, and then similarly for other U_i?

We are thinking of φ_t as the one-parameter group of diffeomorphisms associated to a "gradient vector field" of H (a pseudogradient vector field in the sense of Palais [16] would be more precise), the structurally stable systems as open sets in the stable manifolds of the local minima of H, and U as these stable manifolds.

We think that a positive answer to this conjecture would be very important, as difficult and optimistic as it seems. Of course, one could just as well restrict one's attention to a single isotopy class in $\mathrm{Diff}^r(M)$.

One might attack this conjecture something like this: Let $\Gamma^0(M)$ be the C^0-sections of M, and $f_\#\colon \Gamma^0(M) \circlearrowleft$ be the map $\sigma \to Df\sigma f^{-1}$. All the known examples where f is structurally stable coincide with $I - f_\#$ being surjective. Define $WS(f_\#)$ the weak spectrum of $f_\#$ to be the points λ where $\lambda I - f_\#$ fails to be surjective. Now $-d(WS(f_\#), 1)$ is negative exactly where $I - f_\#$ is surjective, and one might try to smooth this function. While a result in this direction would be interesting, it would be much more interesting if a positive solution were found which gave some information about the changes in the orbit structure of $\varphi_t(x)$ as t increases, say in terms of usual bifurcation theory. It might be more reasonable to take a Sobolev space of diffeomorphisms, instead of $\mathrm{Diff}^r(M)$, where one could construct an actual gradient function.†

Before going on to consider other stability properties we would like to talk about the main known generic properties. The first theorem in this direction that we know of was proven by Kupka [11] and Smale [31].

Definition $f \in \mathrm{Diff}^r(M)$ is Kupka–Smale if and only if:

1. every periodic point of f is hyperbolic.
2. for any two periodic points p and q of f, $W^u(p)$ and $W^s(q)$ only intersect transversally.

Theorem (Kupka [11] and Smale [31]) The Kupka–Smale diffeomorphisms are generic in $\mathrm{Diff}^r(M)$.

† In fact, the existence of the one-parameter group does follow from general considerations in this case and by itself is not too interesting.

The next two theorems were proven by Pugh only for $\mathrm{Diff}^1(M)$. The question of the validity of these theorems for arbitrary r seems a delicate but very important problem.

Theorem (Pugh [22]) Let $f \in \mathrm{Diff}^r(M)$, and let x be a recurrent point of f, i.e., $x \in \alpha(x) \cup \omega(x)$, then there is an arbitrarily small C^1 perturbation of f, g, such that x is a periodic point of g.

Theorem (Pugh [23]) $\Omega(f) = \overline{\mathrm{Per}(f)}$ is generic in $\mathrm{Diff}^1(M)$.

The first theorem is called the C^1 closing lemma. So there are really two problems.

Problem (1) Is the C^r-closing lemma true? (2) Does $\Omega(f) = \overline{\mathrm{Per}(f)}$ generically in $\mathrm{Diff}^r(M)$?

Recently Takens [39] has proved some genericity properties along the lines of Zeaman's tolerance stability conjecture. In particular, he improves on a result of Pugh [23] and proves the following theorem.

Theorem (Takens [39]) There exists a generic set R in $\mathrm{Diff}^r(M)$ such that for $\forall f$ in R and $E > 0$ there exists a neighborhood U of f in $\mathrm{Diff}^r(M)$ such that if $g \in U$ then $\Omega(f) \subset U_E(\Omega(g))$ and $\Omega(g) \subset U_E(\Omega(f))$.

Here $U_E(\Omega(f))$ means the E-neighborhood of $\Omega(f)$ in M. Of course, one may combine all these generic properties to describe one generic set of diffeomorphisms.

These are some things which we have skipped over in our presentation. First there is the case of Anosov diffeomorphisms, which historically preceded Axiom A diffeomorphisms.

Definition $f \in \mathrm{Diff}^r(M)$ is Anosov if f is hyperbolic on all of M.

Theorem (Anosov [2]) Anosov diffeomorphisms are structurally stable.

The astute reader will notice that this theorem has not been subsumed by later theorems, because Robbin's theorem assumes that f is C^2, while on the other hand the Ω-stability theorem may not cover this case because we have not asserted that $\Omega = M$.

However, there is the following still open problem.

Problem If $f \in \mathrm{Diff}^r(M)$ is Anosov, does $\Omega(f) = M$?

In the case where E^u or E^s is a line bundle over M, Newhouse [14] has answered this question positively. Some other problems about Anosov diffeomorphisms stated in [34] and still unresolved are:

Problem Does every Anosov diffeomorphism of a compact manifold have a fixed point? and if M admits an Anosov diffeomorphism, is the universal covering space of M diffeomorphic to R^n? Warren White [40] at this conference has found a complete metric on R^2 with an Anosov-diffeomorphism with no periodic point.

We will not go into these problems, or the problem of finding all Anosov diffeomorphisms up to topological conjugacy. Once again we emphasize that this work is far from complete. Notably, we have excluded the large amount of analysis that has gone into studying the structure of the Ω_is in Smale's theorem, and the work done on the ergodic properties of these Ω_is and Anosov diffeomorphisms, etc.

Returning to stability properties, we want to start by pointing out that the spectral decomposition and no-cycle diffeomorphisms describe a wider class of diffeomorphisms than the Axiom A and no-cycle diffeomorphisms. In fact, large open sets of such examples are constructed in [9] and [25]; these examples have a stability property which is defined below.

Definition $f \in \mathrm{Diff}^r(M)$ is topologically Ω-stable if there exists an open set U of f in $\mathrm{Diff}^r(M)$ such that for all $g \in U$, $\Omega(g)$ is homeomorphic to $\Omega(f)$.

It is still not known if the topologically Ω-stable diffeomorphisms are generic im $\mathrm{Diff}^r(M)$. Although this is probably false, the topologically Ω-stable diffeomorphisms are interesting. The trouble with asking if they are generic right now is that in order to verify the conjecture one would have to know all the perturbations of most diffeomorphisms. The study of perturbations of Lyapunov functions, although far away, might give some information about this. Still we state:

Problem Are the topologically Ω-stable diffeomorphisms generic?

Finally, we would like to end by calling attention to Sotomayor's work on bifurcations and his notion of k-stable diffeomorphisms, which may be found in these proceedings, see [38]. It would seem that the classification of these diffeomorphisms would be very important from the point of view of catastrophe theory which received so much attention at this conference.

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Bounded Orbits in Mechanical Systems with Two Degrees of Freedom and Symmetry†‡

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I Introduction

Smale [6] studied geometrically the problem of describing the flow in a mechanical system of two degrees of freedom with symmetry. In this situation, the group S^1 acts on a two-dimensional orientable manifold M (configuration space), with the action leaving invariant a potential function $V: M \to \mathscr{R}$ and a kinetic energy function (riemannian metric on M) $K: TM \to \mathscr{R}$. The flow of the system is defined on TM (phase space) by the energy $E \equiv K + V \circ \pi: TM \to \mathscr{R}$ and Hamilton's equations, where $\pi: TM \to M$ is the projection. (For a general reference, see [1, 3, or 6].) Both the energy function E and the angular momentum function $J: TM \to \mathscr{R}$ (defined in general in [6]) are integrals of the motion, i.e., are constant on orbits. Let

$$I_{c,p} = E^{-1}(c) \cap J^{-1}(p) = (E \times J)^{-1}(c, p) \subset TM.$$

If the action is free and (c, p) is a regular value of $E \times J$, $I_{c,p}$ is a "cylinder"

† AMS (MOS) 1970 Subject Classification: 58F05, 53A99.
‡ Supported in part by NSF Grant GP-8007.
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