Found Comput Math
DOI 10.1007/s10208-007-9018-5

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

Complexity of Bezout's Theorem VII: Distance Estimates in the Condition Metric

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Received: 12 March 2007 / Final version received: 28 September 2007 / Accepted: 6 November 2007

Abstract We study geometric properties of the solution variety for the problem of approximating solutions of systems of polynomial equations. We prove that given the two pairs (f_i, ζ_i), $i = 1, 2$, there exist a short path joining them such that the complexity of following the path is bounded by the logarithm of the condition number of the problems.

Keywords Approximate zero · Homotopy method · Condition metric

Mathematics Subject Classification (2000) Primary 65H10 · 65H20

1 Introduction

The goal of this paper is to contribute to the search for approximate zeros of systems of polynomial equations. The complexity of homotopy (or path following or continuation) methods for solving systems of polynomial equations has been studied at least since the 1980s (see [7] and references therein, and the series of articles [10–12]). For a survey of complexity results concerning solutions of polynomials of one variable, see [6]. Homotopy methods themselves have a longer history which we do not

Communicated by Felipe Cucker.

Research was partially supported by MTM2007-62799 and by an NSERC discovery grant.

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attempt to survey here. In [13, 14], linear homotopy methods were studied in depth. The existence of a method that finds approximate zeros of systems in average polynomial time was proved, although the lack of specific initial pairs made this proof nonconstructive. A uniform algorithm was not proven to exist (see [15]). A great deal of progress in this direction has recently been made in [2, 4], where the existence of efficient initial pairs for linear homotopies is proved, as well as a probabilistic method to generate them. We refer the reader to [3] for a detailed historical description of the problem and its various solutions.

In [9], a new bound for the complexity of (not necessarily linear) path following was given in terms of the length of the path in the condition metric, which is defined below. In this paper, we prove that there exists surprisingly short paths in the solution variety. Combination of these results suggests the existence of an algorithm that finds approximate zeros of systems very fast, in time almost linear in the size of the input, on the average. It suggests that understanding the geometry of the solution variety in the condition metric, and especially the geodesics may be worth the effort. In Sect. 2, we throw out an idea for a numerical method that the proof of our main result suggests.

For a list of positive degrees $(d)=(d_1,\dots,d_n)\in\mathbb{N}^n$, let $\mathcal{H}_{(d)}$ be the set of all systems $f=(f_1,\dots,f_n)$ of homogeneous polynomials of respective degrees $\deg(f_i)=d_i$, $1\le i\le n$. So, $f:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}^n$. We denote by $D=\max\{d_i:1\le i\le n\}$ the maximum of the degrees. We consider $\mathcal{H}_{(d)}$ endowed with the Bombieri–Weyl Hermitian product, and the corresponding norm (denoted $\|\cdot\|$).

The solution variety $V_{(d)}\subseteq\mathbb{P}(\mathcal{H}_{(d)})\times\mathbb{P}(\mathbb{C}^{n+1})$ (or simply V when there is no possible confusion) is defined as the set of pairs (f,ζ), such that $f(\zeta)=0$. Observe that $V_{(d)}$ is endowed with a natural metric (and corresponding volume form) inherited from the Bombieri–Weyl Hermitian product in $\mathcal{H}_{(d)}$ and the usual Fubini–Study metric in $\mathbb{P}(\mathbb{C}^{n+1})$. We refer to this volume form in $V_{(d)}$ as the Fubini–Study volume.

Let $g\in\mathbb{P}(\mathcal{H}_{(d)})$ be the following system of homogeneous equations (conjectured in [13] to be an efficient initial pair for homotopy methods):

$$g=\begin{cases} d_1^{1/2}X_0^{d_1-1}X_1=0,\\ \quad\vdots\\ d_n^{1/2}X_0^{d_n-1}X_n=0.\end{cases}$$

Observe that $\|g\|=\sqrt{n}$. Moreover, g has a trivial solution $e_0=(1,0,\dots,0)$. In [9], we have bounded the number $k\ge 0$ of steps of projective Newton's method sufficient to follow a homotopy $\Gamma_t=(f_t,\zeta_t)$ in the solution variety V by the length of the path Γ_t in the condition metric,

$$\operatorname{Length}(\Gamma_t)=\int \big\|(\dot f_t,\dot\zeta_t)\big\|_\kappa\,\mathrm{d}t,$$

where $\|(\dot f_t,\dot\zeta_t)\|_\kappa=\mu_{\mathrm{norm}}(f_t,\zeta_t)\|(\dot f_t,\dot\zeta_t)\|$, and μ_{norm} is as in [9, 10]. Namely,

$$\mu_{\mathrm{norm}}(f,\zeta)=\|f\|\,\big\|Df(\zeta)|_{\zeta^\perp}^{-1}\operatorname{Diag}\big(\|\zeta\|^{d_i-1}d_i^{1/2}\big)\big\|,\quad \forall f\in\mathbb{P}(\mathcal{H}_{(d)}),\ \zeta\in\mathbb{P}_n(\mathbb{C}).$$

Then $k\le C_1D^{3/2}\operatorname{Length}(\Gamma_t)$ for some universal constant $C_1>0$. In this paper, we find a short path joining any two pairs in V. Namely, we prove the following result.

Theorem 1 (Main result) *For every pair* $(f,\zeta)\in V_{(d)}$, *such that* $\mu_{\mathrm{norm}}(f,\zeta)<\infty$, *there exists a curve* $\Gamma_t\subseteq V_{(d)}$ *joining* (f,ζ) *and* (g,e_0), *and such that*

$$\mathrm{Length}(\Gamma_t)\le cnD^{3/2}+2\sqrt{n}\ln\left(\frac{\mu_{\mathrm{norm}}(f,\zeta)}{\sqrt{n}}\right),$$

where $c<9$ *is a universal constant.*

Corollary 1 *For every two pairs* $(f,\zeta),(h,\eta)\in V_{(d)}$, *such that* $\mu_{\mathrm{norm}}(f,\zeta)$, $\mu_{\mathrm{norm}}(h,\eta)<\infty$, *there exists a curve* $\Gamma_t\subseteq V_{(d)}$ *joining* (f,ζ) *and* (h,η), *and such that*

$$\mathrm{Length}(\Gamma_t)\le 2cnD^{3/2}+2\sqrt{n}\ln\left(\frac{\mu_{\mathrm{norm}}(f,\zeta)\mu_{\mathrm{norm}}(h,\eta)}{n}\right).$$

Corollary 2 *A sufficient number of projective Newton steps to follow some path in* V *starting at* (g,e_0) *to find an approximate zero associated to a solution* ζ *of a given system* $f\in\mathbb{P}(\mathcal{H}_{(d)})$ *is*

$$C_1D^{3/2}\left(nD^{3/2}+\sqrt{n}\ln\left(\frac{\mu_{\mathrm{norm}}(f,\zeta)}{\sqrt{n}}\right)\right),$$

where C_1 *is a universal constant.*

The real case (i.e., the study of real solutions to real systems of equations) can be analyzed with similar techniques. In this case, the subset of $V_{(d)}$ where μ_{norm} is finite (denoted $W_{(d)}$ or W later in this manuscript) may have one or two connected components, depending on n and (d). Then in each of these connected components Corollaries 1 and 2 hold, with the orthogonal group replacing the unitary group. This observation was also pointed out to us by [5].

The Riemannian metric $\|\cdot\|_k$ defines a metric d_k on $W=V-\{(f,\zeta)\mid\mu_{\mathrm{norm}}(f,\zeta)=\infty\}$ by $d_k(x,y)=\inf\mathrm{Length}(\gamma)$ over piecewise differentiable paths γ in W joining x to y.

Corollary 3 *Let* N *be the dimension of* $\mathcal{H}_{(d)}$. *The probability* (*for the Fubini–Study volume defined above*) *that a pair* $(f,\zeta)\in V$ *belongs to a ball for the condition metric* d_k *of radius* $9nD^{3/2}+\sqrt{n}(4+\ln N+\ln\frac{1}{\varepsilon})$ *centered at* (g,e_0) *is at least*

$$1-\varepsilon.$$

So on the average in V, *a sufficient number of projective Newton steps to follow some path in* W *starting at* (g,e_0) *to find an approximate zero associated to* $(f,\zeta)\in V$ *is less than or equal to* $\tau(n,D,N)$ *where* $\tau(n,D,N)=C_1nD^3\ln N$.

This last corollary suggests that the average number of steps to solve polynomial systems of equations might be $\mathrm{O}(nD^3\ln N)$. The reader may compare this to the result in [13] which suggests that this number might be $\mathrm{O}(nN^3\ln D)$, or to the result in [2, 4] where an upper bound to the average number of steps of $\mathrm{O}(n^5N^2D^4)$ is proved.

The theorem and corollaries above are a consequence of the two following technical propositions, which will be proved in Sect. 4.

Proposition 1 *Let $(f,\zeta)\in V_{(d)}$ be such that $\mu_{\mathrm{norm}}(f,\zeta)<\infty$, and let $U\in\mathcal{U}_{n+1}$ be a unitary matrix such that $Ue_0=\zeta$. Then there exists a unitary matrix $R\in\mathcal{U}_{n+1}$, such that $Re_0=e_0$, and a curve $\Gamma_t\subseteq V_{(d)}$ joining (f,ζ) and $(g\circ R\circ U^*,\zeta)$ and such that*

$$\mathrm{Length}(\Gamma_t)\le 2\sqrt{n}\left(1+\ln\frac{\mu_{\mathrm{norm}}(f,\zeta)}{\sqrt{n}}\right).$$

Proposition 2 *Let U be a unitary matrix, and $\zeta=U^*e_0$. Then there exists a curve $\Gamma_t\subseteq V_{(d)}$ joining (g,e_0) and $(g\circ U,\zeta)$, and such that*

$$\mathrm{Length}(\Gamma_t)\le 2\pi nD^{3/2}.$$

Moreover, we can write $\Gamma_t=(g\circ U_t,U_t^*e_0)$ for a path of unitary matrices $U_t\in\mathcal{U}_{n+1}$.

Assuming Propositions 1 and 2, we can prove the main results of this paper.

1.1 Proof of the Main Results

We start with Theorem 1. We denote by Γ_t^1 the curve that exists from Proposition 1, such that

$$\Gamma_0^1=(f,\zeta),\qquad \Gamma_1^1=(g\circ R\circ U^*,\zeta),$$
$$\mathrm{Length}\big(\Gamma_t^1\big)\le 2\sqrt{n}\left(1+\ln\frac{\mu_{\mathrm{norm}}(f,\zeta)}{\sqrt{n}}\right),$$

where $R,U\in\mathcal{U}_{n+1}$ are unitary matrices, and $Ue_0=\zeta$. Now from Proposition 2, we can join $(g\circ R\circ U^*,\zeta)$ and (g,e_0) with a curve Γ_t^2 of length bounded by $2\pi nD^{3/2}$. Theorem 1 follows. Corollary 1 is clear from Theorem 1.

Corollary 2 is immediate from Theorem 1 and the main theorem of [9]. Finally, we prove Corollary 3. From Theorem 1, we know that

$$\begin{aligned}&\mathrm{Prob}_{(f,\zeta)\in V}\big[\mathrm{dist}_\kappa\big((f,\zeta),(g,e_0)\big)\ge R\big]\\&\quad\le \mathrm{Prob}_{(f,\zeta)\in V}\left[\mu_{\mathrm{norm}}(f,\zeta)\ge\sqrt{n}\exp\left(\frac{R-9nD^{3/2}}{2\sqrt{n}}\right)\right],\end{aligned}$$

for any $R\ge 0$. From Theorem B of [11], this is at most

$$\frac{25N}{\big(\exp\big(\frac{R-9nD^{3/2}}{2\sqrt{n}}\big)\big)^2}.$$

The corollary follows taking

$$R=9nD^{3/2}+\sqrt{n}\left(4+\ln N+\ln\frac{1}{\varepsilon}\right).$$

2 Suggested Numerical Methods

The proof of the main theorem in this paper suggests the following numerical procedure:

(1) INPUT: A polynomial system $f \in \mathcal{H}_{(d)}$.
(2) Let $g' = g$ be the initial system defined in the Introduction, and let $z = e_0$. While z is not an approximate zero of f do:

- For some small $t > 0$ ($t \sim 1/(nD^{3/2})$ might work), let $h = (1-t)g' + tf$ be this polynomial system. Let $z = N_h(z)$, where N_h is projective Newton's operator (cf. [8]).
- Choose a unitary matrix $R \in \mathcal{U}_{n+1}$ such that $\|Rz - e_0\|$ is small. Define $g' = g \circ R$.

(3) OUTPUT: An approximate zero $z \in \mathbb{P}_n(\mathbb{C})$ of f.

There are several ways that the matrix R inside the loop might be chosen. We may choose it at random, or as the result of some Gram–Schmidt procedure. Another suggested way is $R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right) V^*$, where $U(0\ D)V^*$ is a singular value decomposition of the matrix $\mathrm{Diag}(d_i^{-1/2} Dh(z))$.

3 Bundles, Projections

In this section, we prove some technical statements that will be useful for the proof of Propositions 1 and 2. We also include other results that are not necessary for the main results of this paper, but may help to understand the geometry and condition metric in the complex variety V. We consider the following subset of V:

$$W = W_{(d)} = \{(f, \zeta) \in V : Df(\zeta) \text{ is surjective}\}.$$

As in [9], we denote by $\hat{V}$ the affine counterpart of V. Namely,

$$\hat{V} = \{(f, \zeta) \in (\mathcal{H}_{(d)} \setminus \{0\}) \times \mathbb{C}^{n+1} : f(\zeta) = 0\}.$$

As usual, $t \in [0, 1]$ is a parameter, and given a C^1 function $h : [0, 1] \longrightarrow M$ into a manifold M, we may write h_t instead. We also write $\dot{h}_t = Dh(t)(1)$.

We define the "linear" subbundle $\mathcal{L}_{(d)} \subseteq V$ as the set of pairs of the form $(f, \zeta) \in V$, such that $f = (f_1, \dots, f_n)$ and

$$f_i(z) = \left(\frac{\langle z, \zeta \rangle}{\|\zeta\|^2}\right)^{d_i - 1} L_i z,$$

where $L = (L_1, \dots, L_n) : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ is a surjective linear map, such that $L\zeta = 0$. We denote by $\tilde{\mathcal{L}}_{(d)} \subseteq \mathbb{P}(\mathcal{H}_{(d)}) \times \mathbb{S}^{2n+1}$ the corresponding concept when the solutions are in the sphere $\mathbb{S}^{2n+1}$. Finally, the corresponding affine concept will be denoted by $\hat{\mathcal{L}}_{(d)}$. Namely, $\hat{\mathcal{L}}_{(d)}$ is the set of pairs of the form $(f, \zeta) \in \hat{V}$ such that

$f = (f_1, \dots, f_n)$ and

$$f_i(z) = \left(\frac{\langle z, \zeta \rangle}{\|\zeta\|^2} \right)^{d_i - 1} L_i z,$$

where $L = (L_1, \dots, L_n) : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ is a surjective linear map such that $L\zeta = 0$.

For fixed ζ, we consider the set

$$\mathcal{L}_\zeta = \left\{ f \in \mathbb{P}(\mathcal{H}_{(d)}) : (f, \zeta) \in \mathcal{L}_{(d)} \right\}.$$

We also consider the projection $\pi_{\mathcal{L}_{(d)}} : W_{(d)} \longrightarrow \mathcal{L}_{(d)}$, $(f, \zeta) \mapsto (h, \zeta)$ where $h \in \mathbb{P}(\mathcal{H}_{(d)})$ is the system defined as

$$h(z) = \mathrm{Diag} \left(\frac{\langle z, \zeta \rangle}{\|\zeta\|^2} \right)^{d_i - 1} Df(\zeta) z,$$

which can be checked to be well defined. The following property holds for every representative f of a system in $\mathbb{P}(\mathcal{H}_{(d)})$ (see [10]):

$$f = h \oplus h', \quad \text{where } h, h' \in \mathcal{H}_{(d)},\ h' \perp \mathcal{L}_\zeta.$$

In particular, we conclude that $\|f\| \geq \|h\|$. Moreover, the following also holds:

$$Df(\zeta) = Dh(\zeta).$$

We conclude that

$$\mu_{\mathrm{norm}}(f, \zeta) = \frac{\|f\|}{\|h\|} \mu_{\mathrm{norm}}(h, \zeta),$$

where f and h are seen as elements in $\mathbb{P}(\mathcal{H}_{(d)})$.

We also consider the mappings

$$\begin{array}{rccc} \varphi : & \tilde{\mathcal{L}}_{(1)} & \longrightarrow & \tilde{\mathcal{L}}_{(d)} \\ & (L, \zeta) & \mapsto & (f, \zeta), \end{array}$$

where $f \in \tilde{\mathcal{L}}_\zeta$ is defined as

$$f(z) = \mathrm{Diag}\left(d_i^{1/2} \langle z, \zeta \rangle^{d_i - 1} t \right) L z,$$

and

$$\begin{array}{rccc} \phi = \varphi^{-1} : & \tilde{\mathcal{L}}_{(d)} & \longrightarrow & \tilde{\mathcal{L}}_{(1)} \\ & (f, \zeta) & \mapsto & (L, \zeta), \end{array}$$

where $L : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$ is the linear map defined as follows,

$$Lz = \mathrm{Diag}\left(d_i^{-1/2} \right) Df(\zeta) z.$$

Whenever we have a pair (X, Y), we will denote

$$\pi_1(X, Y) = X, \qquad \pi_2(X, Y) = Y.$$

Observe that the following equalities hold for every $(f, \zeta) \in \tilde{\mathcal{L}}_{(d)}$, $(L, \zeta) \in \tilde{\mathcal{L}}_{(1)}$:

$$\mu_{\text{norm}}(f, \zeta) = \mu_{\text{norm}}\big(\phi(f, \zeta)\big), \qquad \mu_{\text{norm}}(L, \zeta) = \mu_{\text{norm}}\big(\varphi(L, \zeta)\big).$$

We will use the following inequality, which holds for every pair of homogeneous polynomials f, g of degrees d_f, d_g (cf. [1])

$$\|fg\| \leq \|f\|\|g\|. \tag{3.1}$$

The following will also be useful. Let f be a homogeneous polynomial of degree d, f defined by

$$f(z) = \langle z, \zeta\rangle^d,$$

where $\zeta \in \mathbb{C}^{n+1}$. Then the following holds:

$$\|f\| = \|\zeta\|^d. \tag{3.2}$$

We will make use of the higher derivative estimate obtained in [10]: For a homogeneous polynomial f of degree d, and for $k \geq 0$,

$$\big\|D^k f(x)(w_1, \ldots, w_k)\big\| \leq d(d-1)\cdots(d-k+1)\|f\|\|x\|^{d-k}\|w_1\|\cdots\|w_k\|, \tag{3.3}$$

for every $x, w_i \in \mathbb{C}^{n+1}$.

For any integer $k \geq 1$ we denote by I_k the identity square matrix of size k.

Lemma 1 *Let $k \geq 1$ and $U \in \mathcal{U}_k$ be a unitary matrix. Then there exists a smooth path $U_t \subseteq \mathcal{U}_k$, $0 \leq t \leq 1$, such that $U_0 = I_k$, $U_1 = U$ and*

$$\text{Length}(U_t) \leq \pi\sqrt{k},$$

where the length is measured for the Frobenius norm.

Proof As U is unitary, it is normal, and hence we can write

$$U = VDV^*,$$

where V is unitary and D is a diagonal matrix containing the eigenvalues of U (this is the well-known Schur decomposition of a normal matrix). Hence, we can write $D = \text{Diag}(\mathrm{e}^{a_1\mathrm{i}}, \ldots, \mathrm{e}^{a_k\mathrm{i}})$ for some real numbers $-\pi \leq a_j \leq \pi$. Now let $A = VD'V^*$ be this skew-symmetric matrix where

$$D' = \text{Diag}(a_1\mathrm{i}, \ldots, a_k\mathrm{i}).$$

We define the path $U_t = \exp(tA)$. Note that $U_0 = I_k$ and $U_1 = \exp(VD'V^*) = V\exp(D')V^* = VDV^* = U$. Moreover,

$$\mathrm{Length}(U_t) = \int_0^1 \|\dot{U}_t\|_F \,\mathrm{d}t = \int_0^1 \|A\|_F \,\mathrm{d}t = \int_0^1 \|D'\|_F \,\mathrm{d}t = \|D'\|_F.$$

Finally, observe that

$$\|D'\|_F^2 = a_1^2 + \cdots + a_k^2 \le \pi^2 k.$$ □

The following lemma is not necessary for the main results of this paper.

Lemma 2 *Let* $f = (f_1, \ldots, f_n) \in \mathcal{H}_{(d)}$ *and* A *be a square matrix of size* $n+1$. *Let* $f' = (f_1', \ldots, f_n') \in \mathcal{H}_{(d)}$ *be defined as*

$$f'(X) = Df(X)(AX), \quad \forall X \in \mathbb{C}^{n+1}.$$

Namely, for $i = 1, \ldots, n$, *we have*

$$f_i'(X) = \left(\frac{\partial f_i}{\partial X_0}(X) \cdots \frac{\partial f_i}{\partial X_n}(X)\right) AX.$$

Then

$$\|f'\| \le n^{3/2} D\|f\|\|A\|_F.$$

Proof Let $f_i = \sum_{|\alpha|=d_i} a_\alpha^i X^\alpha$ be the dense encoding of f_i. Then

$$f_i'(X) = \sum_{k=0}^n h_k^i,$$

where

$$h_k^i(X) = \frac{\partial f_i}{\partial X_k}(X)(AX)_k = \left(\sum_{|\alpha|=d_i} \alpha_k a_\alpha^i X_0^{\alpha_0} \cdots X_k^{\alpha_k - 1} \cdots X_n^{\alpha_n}\right)(AX)_k.$$

From inequality (3.1),

$$\begin{aligned}\|h_k^i\| &\le \left(\sum_{|\alpha|=d_i, \alpha_k \ge 1} \alpha_k^2 \binom{d_i - 1}{\alpha_0 \ldots \alpha_k - 1 \ldots \alpha_n}^{-1} |a_\alpha^i|^2\right)^{1/2} \|A_k\| \\ &= \left(\sum_{|\alpha|=d_i, \alpha_k \ge 1} d\alpha_k \binom{d_i}{\alpha_0 \ldots \alpha_n}^{-1} |a_\alpha^i|^2\right)^{1/2} \|A_k\| \le D\|f\|\|A_k\|,\end{aligned}$$

where $\|A_k\|$ is the norm of the k-th row of A. We conclude that

$$\|f_i'\| \le D\|f\| \sum_{k=0}^n \|A_k\| \le nD\|f\| \left(\sum_{k=0}^n \|A_k\|^2\right)^{1/2} = nD\|f\|\|A\|_F,$$

and the lemma follows. □

Lemma 3 *Let f be a homogeneous polynomial of degree d, and A be a square matrix of size $n+1$. Then*

$$\|f\circ A\| \le \|f\|\|A\|^d,$$

where $f\circ A\in\mathcal{H}_{(d)}$ is the homogeneous polynomial defined by $(f\circ A)(z)=f(Az)$.

Proof First, assume that $A=\mathrm{Diag}(\sigma_0\ge\cdots\ge\sigma_n)$ is a diagonal matrix, with nonnegative entries. Let $f=\sum_{|\alpha|=d}a_\alpha X^\alpha$. Then

$$f\circ A(X)=\sum_{|\alpha|=d}a_\alpha\sigma_0^{\alpha_0}\cdots\sigma_n^{\alpha_n}X^\alpha,$$

and we conclude that

$$\|f\circ A\|^2=\sum_{|\alpha|=d}\binom{d}{\alpha}^{-1}|a_\alpha|^2\sigma_0^{2\alpha_0}\cdots\sigma_n^{2\alpha_n}\le\sigma_0^{2d}\sum_{|\alpha|=d}\binom{d}{\alpha}^{-1}|a_\alpha|^2=\|A\|^{2d}\|f\|^2,$$

and the lemma follows in this case. Now for the general case, let $A=UDV^*$ be a singular value decomposition of A. Then

$$\|f\circ A\|=\|f\circ UDV^*\|=\|f\circ UD\|\le\|f\circ U\|\|D\|^d=\|f\|\|A\|^d,$$

as wanted. □

Lemma 4 *Let $\hat{\psi}_1:\hat{V}\longrightarrow\mathcal{H}_{(d)}$ and $\hat{\psi}_2:\hat{V}\longrightarrow\mathbb{C}^{n+1}$ be two mappings, such that*

$$\big(\hat{\psi}_1(f,\zeta),\hat{\psi}_2(f,\zeta)\big)\in\hat{V},\quad\forall(f,\zeta)\in\hat{V}.$$

Consider the mapping $\hat{\psi}=\hat{\psi}_1\times\hat{\psi}_2:\hat{V}\longrightarrow\hat{V}$, $\hat{\psi}(f,\zeta)=(\hat{\psi}_1(f,\zeta),\hat{\psi}_2(f,\zeta))$. Assume that $\hat{\psi}$ is differentiable, and that the associated mapping $\psi:V\longrightarrow V$ is well defined in some open set containing $(f,\zeta)\in V$. Then

$$\big\|D\psi(f,\zeta)\big\|^2\le\frac{\|D\hat{\psi}_1(f,\zeta)\|^2}{\|\hat{\psi}_1(f,\zeta)\|^2}+\frac{\|D\hat{\psi}_2(f,\zeta)\|^2}{\|\hat{\psi}_2(f,\zeta)\|^2},$$

where some representatives f,ζ of norm equal to 1 have been chosen.

Proof Let f,h and ζ,η be chosen representatives of norm equal to 1, $\hat{\psi}(f,\zeta)=(\alpha h,\beta\eta)$, where

$$\alpha=\big\|\hat{\psi}_1(f,\zeta)\big\|,\qquad\beta=\big\|\hat{\psi}_2(f,\zeta)\big\|.$$

Note that the derivative of

$$\begin{aligned}\pi_f: f^\perp &\mapsto \mathbb{P}(\mathcal{H}_{(d)})\\ \dot{f}&\mapsto f+\dot{f}\end{aligned}$$

is an isometry at 0. The same holds for the (similarly defined) mappings $\pi_h : h^\perp \longrightarrow \mathbb{P}(\mathcal{H}_{(d)})$, $\pi_\zeta : \zeta^\perp \longrightarrow \mathbb{P}(\mathbb{C}^{n+1})$ and $\pi_\eta : \eta^\perp \longrightarrow \mathbb{P}(\mathbb{C}^{n+1})$. Hence,

$$\big\| D\psi(f,\zeta)\big\| = \big\| D(\bar{\psi}_{f,\zeta})(0,0)\big\|,$$

where $\bar{\psi}_{f,\zeta} = (\pi_h \times \pi_\eta)^{-1} \circ \psi \circ (\pi_f \times \pi_\zeta)$ is this mapping between affine spaces. Now, we define the mappings

$$\begin{array}{rcl} \hat{\pi}_f : f^\perp & \mapsto & f + f^\perp, \\ \dot{f} & \mapsto & f + \dot{f} \end{array} \qquad \begin{array}{rcl} \hat{\pi}_\zeta : \zeta^\perp & \mapsto & \zeta + \zeta^\perp, \\ \dot{\zeta} & \mapsto & \zeta + \dot{\zeta}, \end{array}$$

$$\begin{array}{rcl} \Theta_{h,\eta} : \big(\mathcal{H}_{(d)} \setminus h^\perp\big) \times \big(\mathbb{C}^{n+1} \setminus \eta^\perp\big) & \longrightarrow & h^\perp \times \eta^\perp \\ (u,x) & \mapsto & \left(\dfrac{\|h\|^2}{\langle u,h\rangle} u - h, \dfrac{\|\eta\|^2}{\langle x,\eta\rangle} x - \eta \right). \end{array}$$

The derivatives of $\hat{\pi}_f$ and $\hat{\pi}_\zeta$ at 0 are again isometries. Moreover, we can easily check that

$$\big\| D\Theta_{h,\eta}(\alpha h, \beta\eta)(u,x)\big\|^2 \le \frac{\|u\|^2}{\alpha^2} + \frac{\|x\|^2}{\beta^2}.$$

Finally, observe that

$$\bar{\psi}_{f,\zeta} = \Theta_{h,\eta} \circ \hat{\psi} \circ (\hat{\pi}_f \times \hat{\pi}_\zeta).$$

We conclude that

$$\begin{aligned} \big\| D\bar{\psi}_{f,\zeta}(0,0)(\dot{f},\dot{\zeta})\big\|^2 &= \big\| D\Theta_{h,\eta}(\alpha h, \beta\eta)\big(D\hat{\psi}(f,\zeta)(\dot{f},\dot{\zeta})\big)\big\|^2 \\ &\le \frac{\|D\hat{\psi}_1(f,\zeta)(\dot{f},\dot{\zeta})\|^2}{\alpha^2} + \frac{\|D\hat{\psi}_2(f,\zeta)(\dot{f},\dot{\zeta})\|^2}{\beta^2}, \end{aligned}$$

and thus,

$$\big\| D\bar{\psi}_{f,\zeta}(0,0)\big\|^2 \le \frac{\|D\hat{\psi}_1(f,\zeta)\|^2}{\alpha^2} + \frac{\|D\hat{\psi}_2(f,\zeta)\|^2}{\beta^2}.$$

The lemma follows. □

Lemma 5 *Let $\langle\cdot,\cdot\rangle_*$ be any dot product in $\mathbb{R}^{k+1}$ and let $\mathbb{S}^k_*(r) \subseteq \mathbb{R}^{k+1}$ be the radius r sphere for that dot product. Let $a, b \in \mathbb{S}^k_*(r)$ be any two points, $a \ne -b$. Let x_t be the curve*

$$x_t = r\frac{(1-t)a + tb}{\|(1-t)a + tb\|_*} \subseteq \mathbb{S}^k_*(r).$$

Then for any $0 \le t \le 1$,

$$\big\|(1-t)a + tb\|_* \|\dot{x}_t\big\|_* \le 2r^2.$$

Proof Observe that $x_t = \Theta_1 \circ \Theta_2(t)$, where

$$\begin{aligned} \Theta_2 : [0,1] &\longrightarrow \mathbb{R}^{k+1}, & \Theta_1 : \mathbb{R}^{k+1} &\longrightarrow \mathbb{S}_*^k(r), \\ t &\mapsto (1-t)a + tb, & x &\mapsto r\frac{x}{\|x\|}. \end{aligned}$$

Hence,

$$\|\dot{x}_t\|_* \le \big\| D\Theta_1\big(\Theta_2(t)\big)\big\|_* \big\| D\Theta_2(t)\big\|_* = \big\| D\Theta_1\big(\Theta_2(t)\big)\big\|_* \|a-b\|_*.$$

Now,

$$D\Theta_1(x)(v) = \frac{r}{\|x\|_*}\left(v - \frac{\langle v, x\rangle_*}{\|x\|_*^2}x\right).$$

Now, observe that $\frac{\langle v,x\rangle_*}{\|x\|_*^2}x$ is the projection of v onto $\mathrm{Span}(x)$, and we conclude that

$$\big\| D\Theta_1(x)\big\|_* \le \frac{r}{\|x\|_*},$$

so the lemma follows. □

Lemma 6 *The norm of the derivative of* $\pi_{\mathcal{L}_{(d)}}$ *satisfies the following inequality*:

$$\big\| D\pi_{\mathcal{L}_{(d)}}(f,\zeta)\big\| \le \sqrt{3}D^2\frac{\|f\|}{\|h\|},$$

where $(h,\zeta) = \pi_{\mathcal{L}_{(d)}}(f,\zeta)$.

Proof Let f and ζ be chosen representatives, $\|f\| = \|\zeta\| = 1$. We denote by $\hat{\pi}_{\mathcal{L}_{(d)}} : \hat{W}_{(d)} \longrightarrow \hat{\mathcal{L}}_{(d)} \subseteq \hat{W}_{(d)}$ the affine version of the mapping $\pi_{\mathcal{L}_{(d)}}$, and $(h,\zeta) = \hat{\pi}_{\mathcal{L}_{(d)}}(f,\zeta)$, so that $\|h\| \le \|f\| = 1$. Then we are under the assumptions of Lemma 4. Moreover, for $\dot{f} \in f^\perp$ and $\dot{\zeta} \in \zeta^\perp$, we have:

$$D\hat{\pi}_{\mathcal{L}_{(d)}}(f,\zeta)(\dot{f},\dot{\zeta}) = (\dot{h},\dot{\zeta}),$$

where $\dot{h} = (\dot{h}_1,\dots,\dot{h}_n)$ is defined by $\dot{h}_i = p_i + q_i$, $p_i \perp q_i$, and

$$p_i(z) = (d_i-1)\langle z,\zeta\rangle^{d_i-2}\langle z,\dot{\zeta}\rangle Df_i(\zeta)(z),$$

$$q_i(z) = \langle z,\zeta\rangle^{d_i-1}\big(D^{(2)}f_i(\zeta)(\dot{\zeta},z) + D\dot{f}_i(\zeta)(z)\big).$$

We conclude that $\|\dot{h}_i\|^2 = \|p_i\|^2 + \|q_i\|^2$. We estimate each of these two norms separately. From (3.2) and (3.1), we conclude:

$$\|p_i\| \le (d_i-1)\|\dot{\zeta}\|\big\|Df_i(\zeta)\big\| \le (D-1)\|\dot{\zeta}\|\big\|Df_i(\zeta)\big\|.$$

Inequality (3.3) yields

$$\|p_i\| \le D(D-1)\|f_i\|\|\dot{\zeta}\|.$$

On the other hand, again (3.2) and (3.1) imply

$$\|q_i\| \leq \left\| D^{(2)} f_i(\zeta)(\dot{\zeta}) \right\| + \left\| D\dot{f}_i(\zeta) \right\|.$$

Inequality (3.3) yields

$$\|q_i\| \leq D(D-1)\|f_i\|\|\dot{\zeta}\| + D\|\dot{f}_i\|.$$

We conclude that

$$\|\dot{h}\|^2 = \sum_{i=1}^{n} \|\dot{h}_i\|^2 \leq 3D^2(D-1)^2\|\dot{\zeta}\|^2 + 2D^2\|\dot{f}\|^2 \leq 3D^2(D-1)^2\left\|(\dot{f}, \dot{\zeta})\right\|^2.$$

Hence, from Lemma 4,

$$\left\| D\pi_{\mathcal{L}_{(d)}}(f, \zeta) \right\|^2 \leq \frac{3D^2(D-1)^2}{\|h\|^2} + 1 \leq \frac{3D^4}{\|h\|^2}.$$

We have chosen a representative such that $\|f\| = 1$. Now, observe that if we multiply f by $\lambda \in \mathbb{C}^*$ then h is multiplied by the same quantity. The lemma follows. $\square$

Proposition 3 *The following inequalities hold.*

$$\left\| D\varphi(L, \zeta) \right\| \leq D^{3/2}, \tag{3.4}$$

$$\left\| D\phi(f, \zeta) \right\| \leq \sqrt{2} D^{3/2}. \tag{3.5}$$

Proof First we prove inequality (3.4). Observe that

$$D\varphi(L, \zeta)(\dot{L}, \dot{\zeta}) = (\dot{g}, \dot{\zeta}),$$

where $\dot{g} = (\dot{g}_1, \ldots, \dot{g}_n)$ satisfies $\dot{g}_i = p_i + q_i$, $\dot{g}_i \perp \pi_1(\varphi(L, \zeta))$ and

$$p_i(z) = d_i^{1/2}(d_i - 1)\langle z, \zeta\rangle^{d_i - 2}\langle z, \dot{\zeta}\rangle L_i z,$$

$$q_i(z) = d_i^{1/2}\langle z, \zeta\rangle^{d_i - 1}\dot{L}_i z.$$

Moreover, observe that $p_i \perp q_i$. Indeed, by unitary invariance it suffices to prove this in the case that $\zeta = e_0$. Now in this case,

$$p_i(z) = d_i^{1/2}(d_i - 1)z_0^{d_i - 2} h_i(z_1, \ldots, z_n), \; q_i(z) = d_i^{1/2} z_0^{d_i - 1} h_i'(z_1, \ldots, z_n),$$

for some polynomials h_i, h_i'. We conclude that p_i and q_i have no monomials in common, and hence they are orthogonal.

From (3.1) and (3.2),

$$\|p_i\| \le D^{1/2}(D-1)\|\dot\zeta\|\|L_i\|, \qquad \|q_i\| \le D^{1/2}\|\dot L_i\|.$$

We conclude that

$$\|\dot g\|^2 = \sum_{i=1}^{n}\bigl(\|p_i\|^2 + \|q_i\|^2\bigr) \le D(D-1)^2\|\dot\zeta\|^2\|L\|_F^2 + D\|\dot L\|_F^2.$$

Hence,

$$\begin{aligned}
\bigl\|D\varphi(L,\zeta)(\dot L,\dot\zeta)\bigr\|^2 &= \frac{\|\dot g\|^2}{\|\pi_1(\varphi(L,\zeta))\|^2} + \|\dot\zeta\|^2\\
&= \frac{\|\dot g\|^2}{\|L\|_F^2} + \|\dot\zeta\|^2 \le D\frac{(D-1)^2\|\dot\zeta\|^2\|L\|_F^2 + \|\dot L\|_F^2}{\|L\|_F^2} + \|\dot\zeta\|^2\\
&\le D^3\left(\frac{\|\dot L\|_F^2}{\|L\|_F^2} + \|\dot\zeta\|^2\right) = D^3\bigl\|(\dot L,\dot\zeta)\bigr\|^2,
\end{aligned}$$

and (3.4) follows.

Finally, we prove (3.5). Observe that for $(f,\zeta)\in\tilde{\mathcal{L}}_{(d)}$ and $(\dot f,\dot g)\in T_{(f,\zeta)}\tilde{\mathcal{L}}_{(d)}$, we have that

$$D\phi(f,\zeta)(\dot f,\dot\zeta) = (\dot L,\dot\zeta)\in T_{\phi(f,\zeta)}\tilde{\mathcal{L}}_{(1)},$$

where $\dot L = (\dot L_1,\dots,\dot L_n)$ is the linear map defined as

$$\dot L_i(z) = d_i^{-1/2}\bigl(D^{(2)}f_i(\zeta)(\dot\zeta,z) + D\dot f_i(\zeta)(z)\bigr).$$

We conclude that

$$\|\dot L_i\|^2 \le \frac{2}{d_i}\bigl(\bigl\|D^{(2)}f_i(\zeta)(\dot\zeta)\bigr\|^2 + \bigl\|D\dot f_i(\zeta)\bigr\|^2\bigr).$$

Inequality (3.3) yields

$$\|\dot L_i\|^2 \le 2D(D-1)^2\|f_i\|^2\|\dot\zeta\|^2 + 2D\|\dot f_i\|^2.$$

Thus,

$$\|\dot L\|_F^2 \le 2D(D-1)^2\|f\|^2\|\dot\zeta\|^2 + 2D\|\dot f\|^2.$$

We conclude that

$$\begin{aligned}
\bigl\|D\phi(f,\zeta)(\dot f,\dot\zeta)\bigr\|^2 &= \frac{\|\dot L\|^2}{\|\pi_1(\phi(f,\zeta))\|_F^2} + \|\dot\zeta\|^2\\
&\le \frac{2D(D-1)^2\|f\|^2\|\dot\zeta\|^2 + 2D\|\dot f\|^2}{\|\operatorname{Diag}(d_i^{-1/2})Df(\zeta)\|_F^2} + \|\dot\zeta\|^2.
\end{aligned}$$

On the other hand, observe that if $f \in \tilde{\mathcal{L}}_{(d)}$, the following equality holds:

$$\|f\| = \left\|\mathrm{Diag}\left(d_i^{-1/2}\right) Df(\zeta)\right\|_F.$$

We conclude that

$$\begin{aligned}\left\|D\phi(f,\zeta)(\dot{f},\dot{\zeta})\right\|^2 &\leq 2D\frac{\|\dot{f}\|^2}{\|f\|^2} + \left(2D(D-1)^2+1\right)\|\dot{\zeta}\|^2 \\ &\leq 2D^3\left(\frac{\|\dot{f}\|^2}{\|f\|^2} + \|\dot{\zeta}\|^2\right) = 2D^3\left\|(\dot{f},\dot{\zeta})\right\|^2,\end{aligned}$$

and inequality (3.5) follows. □

Corollary 4 *Let Γ_t be a curve in $\tilde{\mathcal{L}}_{(1)}$, $t \in [0,1]$. Then*

$$\mathrm{Length}\left(\varphi(\Gamma_t)\right) \leq D^{3/2}\,\mathrm{Length}(\Gamma_t).$$

Now, let Γ_t be a curve in $\tilde{\mathcal{L}}_{(d)}$, $t \in [0,1]$. Then

$$\mathrm{Length}\left(\phi(\Gamma_t)\right) \leq \sqrt{2}D^{3/2}\,\mathrm{Length}(\Gamma_t).$$

Finally, let Γ_t be a curve in $W_{(d)}$, $t \in [0,1]$. Then

$$\mathrm{Length}\left(\pi_{\mathcal{L}_{(d)}}(\Gamma_t)\right) \leq \sqrt{3}D^2\,\mathrm{Length}(\Gamma_t).$$

Proof For the first inequality, denote $f_t = \pi_1(\varphi(L_t,\zeta_t))$. Then

$$\begin{aligned}\mathrm{Length}\left(\varphi(\Gamma_t)\right) &= \int_0^1 \mu_{\mathrm{norm}}(f_t,\zeta_t)\left\|(\dot{f},\dot{\zeta})\right\|\,\mathrm{d}t \\ &\leq \int_0^1 \mu_{\mathrm{norm}}(L_t,\zeta_t)\left\|D\varphi(L,\zeta)\right\|\left\|(\dot{L},\dot{\zeta})\right\|\,\mathrm{d}t.\end{aligned}$$

From Proposition 3, this is less than or equal to

$$\int_0^1 \mu_{\mathrm{norm}}(L_t,\zeta_t)D^{3/2}\left\|(\dot{L},\dot{\zeta})\right\|\,\mathrm{d}t = D^{3/2}\,\mathrm{Length}(\Gamma_t),$$

as wanted.

For the second inequality, let $L_t = \pi_1(\phi(f_t,\zeta_t))$, and observe that

$$\begin{aligned}\mathrm{Length}\left(\phi(\Gamma_t)\right) &= \int_0^1 \mu_{\mathrm{norm}}(L_t,\zeta_t)\left\|(\dot{L},\dot{\zeta})\right\|\,\mathrm{d}t \\ &\leq \int_0^1 \mu_{\mathrm{norm}}(f_t,\zeta_t)\left\|D\phi(f,\zeta)\right\|\left\|(\dot{f},\dot{\zeta})\right\|\,\mathrm{d}t.\end{aligned}$$

From Proposition 3, this is less than or equal to

$$\int_0^1 \mu_{\text{norm}}(f_t,\zeta_t)\sqrt{2}D^{3/2}\|(\dot{f},\dot{\zeta})\|\,\mathrm{d}t=\sqrt{2}D^{3/2}\,\text{Length}(\Gamma_t).$$

The third inequality is proved in the very same way, using Lemma 6 instead of Proposition 3. □

4 Proof of Propositions 1 and 2

4.1 Proof of Proposition 1

First, assume that $\zeta=e_0$.

We choose a representative of f, such that $\|f\|=\sqrt{n}$. As $f(e_0)=0$, the matrix $\text{Diag}(d_i^{-1/2})Df(e_0)$ may be written as

$$\text{Diag}\big(d_i^{-1/2}\big)Df(e_0)=\big(0\ \bar{U}D\bar{V}^*\big),$$

where $\bar{U},\bar{V}\in\mathcal{U}_n$ are unitary matrices, and $D=\text{Diag}(\sigma_1\geq\cdots\geq\sigma_n>0)$ is a diagonal matrix with real positive entries. Moreover, as $\mu_{\text{norm}}(f,\zeta)\geq\sqrt{n}$ always,

$$\sqrt{n}\leq\mu_{\text{norm}}(f,\zeta)=\frac{\|f\|}{\sigma_n}=\frac{\sqrt{n}}{\sigma_n},$$

and we conclude that $\sigma_n\leq 1$. We denote

$$R=\begin{pmatrix}1 & 0\\ 0 & \bar{U}\bar{V}^*\end{pmatrix}\in\mathcal{U}_{n+1}.$$

Observe that

$$\text{Diag}\big(d_i^{-1/2}\big)D(g\circ R)(e_0)=\big(0\ \bar{U}\bar{V}^*\big).$$

We define the curve

$$\Gamma_t'=(f_t,e_0)=\left(\sqrt{n}\frac{(1-t)f+tg\circ R}{\|(1-t)f+tg\circ R\|},e_0\right)\subseteq\mathbb{S}_{\sqrt{n}}(\mathcal{H}_{(d)})\times\{e_0\},$$

where $\mathbb{S}_{\sqrt{n}}(\mathcal{H}_{(d)})$ is the radius $\sqrt{n}$ sphere in the space $\mathcal{H}_{(d)}$. Then we define Γ_t as the projection of Γ_t' on $\mathbb{P}(\mathcal{H}_{(d)})\times\{e_0\}$.

From Lemma 5, we know that

$$\big\|(1-t)f+tg\circ R\big\|\,\|\dot{f}_t\|\leq 2n.$$

Moreover, the following equality also holds,

$$\text{Diag}\big(d_i^{-1/2}\big)Df_t(e_0)=\frac{\sqrt{n}}{\|(1-t)f+tg\circ R\|}\big(0\ \bar{U}\big((1-t)D+tI_n\big)\bar{V}^*\big).$$

Hence, the following equality holds,

$$\mu_{\mathrm{norm}}(f_t, e_0) = \|f_t\| \left\| \left(\mathrm{Diag}\left(d_i^{-1/2}\right) Df_t(e_0) \mid_{e_0^\perp}\right)^{-1} \right\| = \frac{\|(1-t)f + tg \circ R\|}{(1-t)\sigma_n + t}.$$

We conclude that

$$\begin{aligned}
\mathrm{Length}(\Gamma_t) &\le \frac{\mathrm{Length}(\Gamma_t')}{\sqrt{n}} = \int_0^1 \mu_{\mathrm{norm}}(f_t, e_0) \frac{\|\dot{f}_t\|}{\sqrt{n}} \, \mathrm{d}t \\
&= \int_0^1 \frac{\|(1-t)f + tg \circ R\| \|\dot{f}_t\|}{((1-t)\sigma_n + t)\sqrt{n}} \, \mathrm{d}t \le \int_0^1 \frac{2\sqrt{n}}{(1-t)\sigma_n + t} \, \mathrm{d}t \\
&= 2\sqrt{n} \frac{\ln \frac{1}{\sigma_n}}{1 - \sigma_n} \le 2\sqrt{n}(1 - \ln \sigma_n) = 2\sqrt{n} \left(1 + \ln \frac{\mu_{\mathrm{norm}}(f, e_0)}{\sqrt{n}}\right).
\end{aligned}$$

For the general case, consider the pair $(f \circ U, U^* \zeta = e_0) \in V_{(d)}$. Then there exists a unitary matrix $R \in \mathcal{U}_n$ and a path $\Gamma_t' \subseteq V_{(d)}$ such that

$$\Gamma_0' = (f \circ U, e_0), \qquad \Gamma_1' = (g \circ R, e_0),$$

$$\mathrm{Length}\left(\Gamma_t'\right) \le 2\sqrt{n} \left(1 + \ln \frac{\mu_{\mathrm{norm}}(f \circ U, e_0)}{\sqrt{n}}\right) = 2\sqrt{n} \left(1 + \ln \frac{\mu_{\mathrm{norm}}(f, \zeta)}{\sqrt{n}}\right).$$

We just consider the path $\Gamma_t = (f_t, \zeta)$, where

$$f_t = f_t' \circ U^*.$$

4.2 Proof of Proposition 2

First, assume that $(d) = (1)$. Then $g = (0 \ I_n)$. Let U_t be a curve in $\mathcal{U}_{n+1}$ such that $U_0 = I_n$ and $U_1 = U$. Then we consider the curve

$$\Gamma_t = (g \circ U_t, U_t^* e_0) \subseteq \tilde{\mathcal{L}}_{(1)}.$$

The following holds.

$$\begin{aligned}
\mathrm{Length}(\Gamma_t) &= \int_0^1 \mu_{\mathrm{norm}}\left(g \circ U_t, U_t^* e_0\right) \sqrt{\frac{\|g \circ \dot{U}_t\|_F^2}{\|g \circ U_t\|_F^2} + \left\|\dot{U}_t^* e_0\right\|^2} \, \mathrm{d}t \\
&\le \sqrt{2n} \int_0^1 \|\dot{U}_t\|_F \, \mathrm{d}t = \sqrt{2n} \, \mathrm{Length}(U_t).
\end{aligned}$$

From Lemma 1, we can choose U_t such that $\mathrm{Length}(U_t) \le \pi\sqrt{n+1}$. Finally, this curve in $\tilde{\mathcal{L}}_{(1)}$ can be projected into $\mathcal{L}_{(1)}$, and the proposition follows in the case that $(d) = (1)$. Now for the general case, let $\phi(g, e_0) = ((0 \ I_n), e_0) \in \hat{\mathcal{L}}_{(1)}$ and $(L', \zeta) = \phi(g \circ U, \zeta) \in \hat{\mathcal{L}}_{(1)}$. Observe that $L' = (0 \ I_n)U$. Hence, there exists a curve $\Gamma_t \subseteq \hat{\mathcal{L}}_{(1)}$ joining $\phi(g, e_0)$ and $\phi(g \circ U, \zeta)$, and such that

$$\mathrm{Length}(\Gamma_t) \le 2\pi n.$$

Now from Corollary 4, the curve $\varphi(\Gamma_t) \subseteq \tilde{\mathcal{L}}_{(d)}$, joining (g, e_0) and $(g \circ U, e_0)$, has length bounded by $2\pi n D^{3/2}$, and so has its projection into $\mathcal{L}_{(d)}$.

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