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Dynamics of two-dimensional Blaschke products

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For Bill Parry, In Memoriam

Abstract. In this paper we study the dynamics on $\mathbb{T}^2$ and $\mathbb{C}^2$ of a two-dimensional Blaschke product. We prove that in the case when the Blaschke product is a diffeomorphism of $\mathbb{T}^2$ with all periodic points hyperbolic then the dynamics is hyperbolic. If a two-dimensional Blaschke product diffeomorphism of $\mathbb{T}^2$ is embedded in a two-dimensional family given by composition with translations of $\mathbb{T}^2$, then we show that there is a non-empty open set of parameter values for which the dynamics is Anosov or has an expanding attractor with a unique SRB measure.

1. *Introduction*

A (finite) Blaschke product is a map of the form

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

where $n \geq 2$, $a_i \in \mathbb{C}$, $|a_i| < 1$, $i = 1 \ldots n$ and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$. B is a rational mapping of $\mathbb{C}$, it is an analytic function in a neighborhood of the closed unit disc $\mathbb{D}$, and B maps the unit circle $\mathbb{T}$ to itself. Blaschke products are interesting in their own right but also from the point of view of more general complex dynamics. For example, from the Riemann mapping theorem it follows that any meromorphic map such that its Julia set bounds an invariant simply connected neighborhood in $\mathbb{C}$ is conjugate to a Blaschke product which is expanding on the unit circle [**CG**].

In [**PRS**] we studied the family of Blaschke products $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$, where $B_\theta := \theta B$, as dynamical systems on $\mathbb{T}$ from the point of view of finding lower bounds for the average entropy of members of the family. In [**PRS**] we first show that either B_θ has a fixed sink or indifferent point or the dynamics is expanding. We show that, for an open set U of θ, B_θ is expanding. Then we give a lower bound for the integral of the entropy $h(B_\theta)$ over U, where the entropy is the measure theoretic entropy with respect to the absolutely continuous invariant measure for the expanding B_θ. This measure is also the SRB measure for B_θ.

In the present brief paper we make a start on the study of the dynamics of a two-dimensional Blaschke product of the form

$$F(z, w) = (A(z)B(w), C(z)D(w)), \tag{1.1}$$

and of the form

$$F(z, w) = \left(\frac{A(z)}{B(w)}, \frac{D(w)}{C(z)}\right), \tag{1.2}$$

which are considered as dynamical systems on $\mathbb{T} \times \mathbb{T}$ where A, B, C, D are one-dimensional Blaschke products and we allow the possibility that some of the degrees of A, B, C, D may also be one. We concentrate our study on the case of Blaschke products that are diffeomorphisms of $\mathbb{T}^2$. We will call the Blaschke products of type (1.1) that are diffeomorphisms of $\mathbb{T}^2$ *Blaschke product diffeomorphisms*; the Blaschke products of type (1.2) that are diffeomorphisms of $\mathbb{T}^2$ will be called *quotient Blaschke product diffeomorphisms*. If the periodic points of a Blaschke product diffeomorphism are hyperbolic the diffeomorphism is hyperbolic (see Theorem 3.10). Moreover, some are Anosov diffeomorphisms. In some cases we show that the Julia set of a Blaschke product considered as a map of $\mathbb{C}^2$ is contained in $\mathbb{T}^2$. Observe that the result stated in Theorem 3.10 is similar to that obtained for one-dimensional Blaschke product endomorphisms.

These results could help to gain some insight into the dynamics of certain meromorphic maps on $\mathbb{C}^2$.

In [**PRS**], we raise the question of whether similar results as that stated for one-dimensional Blaschke product families might hold for two-dimensional Blaschke product families. The family now depends on two parameters $(\theta, \phi) \in \mathbb{T} \times \mathbb{T}$,

$$F_{(\theta,\phi)}(z, w) = (\theta A(z)B(w), \phi C(z)D(w)).$$

We show in Theorem 4.1 that if F is a diffeomorphism of $\mathbb{T} \times \mathbb{T}$ then there is an open set of parameter values for which there is a SRB measure with positive entropy that is either supported on the whole torus, in which case the diffeomorphism is Anosov, or on a hyperbolic attractor. This is analogous to our result in one dimension. As opposed to [**PRS**] who employ techniques from complex analysis, our proofs rely on results from real dynamics, especially [**PS1, PS2**]. We wonder whether our results on Blaschke products have simpler proofs more along the lines of complex dynamics. The Blaschke product diffeomorphisms we consider as above are precisely the analytic maps on a neighborhood of $\mathbb{D} \times \mathbb{D}$ mapping $\mathbb{D} \times \mathbb{D}$ to itself and $\mathbb{T} \times \mathbb{T}$ diffeomorphically to itself (see [**R**]).

2. *Examples*

The Blaschke product

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

is a degree n, orientation-preserving immersion of the circle. It follows that the two-dimensional Blaschke product diffeomorphisms F we are considering induce isomorphisms of the fundamental group of the torus, $\mathbb{Z} \times \mathbb{Z}$, which given as a matrix are of the form

$$N_F = \begin{bmatrix} n & m \\ k & j \end{bmatrix}$$

with n, m, k, j positive integers that are the degrees of A, B, C, D, respectively. Moreover, $\det(N) = 1$ since F is an orientation-preserving diffeomorphism of $\mathbb{T} \times \mathbb{T}$. From the form of N_F we see that N_F maps the positive orthant into itself and N_F is a hyperbolic linear map. It is sometimes convenient to consider the torus $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$ and to write F additively as follows. Given a Blaschke product $B : \mathbb{C} \to \mathbb{C}$, there exists $b : \mathbb{R} \to \mathbb{R}$ with

$$b' > 0,$$

such that

$$B(e^{2\pi i x}) = e^{2\pi i b(x)}.$$

Now, given $F(z, w) = (A(z)B(w), C(z)D(w))$, let a, b, c, d be the corresponding transformations acting on $\mathbb{R}$. Therefore

$$F(e^{2\pi i x}, e^{2\pi i y}) = (e^{2\pi i (a(x)+b(y))}, e^{2\pi i (c(x)+d(y))}).$$

So, given F we can take the map

$$\hat{F} : \mathbb{T}^2 \to \mathbb{T}^2,$$
$$\hat{F}(x, y) = (\hat{F}_1(x, y), \hat{F}_2(x, y)) = (a(x) + b(y), c(x) + d(y)).$$

Observe that for $(z, w) = (e^{2\pi i x}, e^{2\pi i y}) \in \mathbb{T}^2$ we have

$$F^n(z, w) = (e^{2\pi i \hat{F}_1^n(x,y)}, e^{2\pi i \hat{F}_2^n(x,y)}).$$

The linear part of $\hat{F}$ is the matrix N_F above, i.e. $\hat{F} = N_F + \Theta$ where Θ is doubly periodic. An invariant set Λ for $f : M \to M$ is said to be hyperbolic if it is compact,the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n_{|E^s(x)}| \leq C\lambda^n, \quad |Df^{-n}_{|E^u(x)}| \leq C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n. Moreover, a diffeomorphism $f : M \to M$ is said to be Anosov if M itself is a hyperbolic set for f. In the case where $M = \mathbb{T}^n$, and f is Anosov, $L(f) = \mathbb{T}^n$, where $L(f)$ is the limit set of f (the closure of the accumulation points of all backward and forward trajectories) [**M**]. It is not known if this last property is true for all Anosov diffeomorphisms of compact manifolds.

2.1. *Blaschke products induced by a linear Anosov diffeomorphisms.* Now let us start with a matrix $N \in SL(2, \mathbb{Z})$,

$$N = \begin{bmatrix} n & m \\ k & j \end{bmatrix}$$

with n, m, k, j positive integers. Let

$$F_N(z, w) = (z^n w^m, z^k w^j)$$

and observe that F_N is a Blaschke product diffeomorphism. It follows that

$$\hat{F}_N(x, y) = N(x, y) = (nx + my, kx + jy),$$

so $\hat{F}_N$ is a linear Anosov diffeomorphism induced by N on $\mathbb{T}^2$ and $\mathbb{T}^2$ is a hyperbolic set for F_N.

If we take

$$N^{-1} = \begin{bmatrix} j & -m \\ -k & n \end{bmatrix}$$

it follows that

$$F_{N^{-1}}(z, w) = (z^j w^{-m}, z^{-k} w^n)$$

is a quotient Blaschke product diffeomorphism.

From the discussion above we see that F is in the homotopy class of the linear Anosov diffeomorphism N_F. So Question 4.2 of §4 makes sense from the homotopy point of view.

2.1.1. *The Dynamics in $\mathbb{C}^2$ of Blaschke products induced by a linear Anosov diffeomorphism.* Now we discuss the dynamics of F_N in $\mathbb{C}^2$.

LEMMA 2.1. $\Omega(F_N) = \{(0, 0)\} \cup \mathbb{T}^2\}$ *and F_N is an Axiom A diffeomorphism acting on* $\mathbb{C}^2 \setminus \{\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}\}$.

Proof. Writing z and w in polar coordinates, $z = r\exp(2\pi ix)$, $w = s\exp(2\pi iy)$, it follows that

$$F_N(r\exp(2\pi ix), s\exp(2\pi iy)) = (r^n s^m \exp 2\pi i(nx + my), r^k s^j \exp 2\pi i(kx + jy)).$$

Studying the map $(r, s) \to (r^n s^m, r^k s^j)$ for $r, s \geq 0$, it follows that its non-wandering set consists of two points: $(0, 0)$, $(1, 1)$. □

The stable manifold of $\mathbb{T}^2$ is contained in $\{|z| < 1\} \times \{|w| > 1\} \cup \{|z| > 1\} \times \{|w| < 1\}$ and the unstable manifold of $\mathbb{T}^2$ is contained in $\{|z| < 1\} \times \{|w| < 1\} \cup \{|z| > 1\} \times \{|w| > 1\}$. The map F_N fails to be a diffeomorphism at $\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$ which are critical points for F_N.

2.1.2. *Perturbation of Blaschke products induced by a linear Anosov diffeomorphism.* The next theorem follows from classical results on the stability of Axiom A systems.

THEOREM 2.2. *Let F_N be a Blaschke product diffeomorphism induced by $N \in Sl(2, \mathbb{Z})$. Then, for any rational map G on $\mathbb{C}^2$ close enough to F_N it follows that $\Omega(G) = \{\{S\} \cup \mathcal{H}\}$, such that S is a fixed attracting point, and $\mathcal{H}$ is a hyperbolic set homeomorphic to a two-dimensional torus. Moreover, if G is a diffeomorphism of $\mathbb{C}^2 \setminus \{\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}\}$ it is an Axiom A diffeomorphism.*

3. *General dynamics of the Blaschke products*

Definition 3.1. An f-invariant set Λ is said to have a dominated splitting, if the tangent bundle over Λ is decomposed into two invariant subbundles $T_\Lambda M = \mathcal{E} \oplus \mathcal{F}$, such that there exist $C > 0$ and $0 < \lambda < 1$ with the following property:

$$|Df^n_{|\mathcal{E}(x)}||Df^{-n}_{|\mathcal{F}(f^n(x))}| \leq C\lambda^n \quad \text{for all } x \in \Lambda,\ n \geq 0.$$

This concept was introduced independently by Mañé, Liao and Pliss, as a first step toward proving that structurally stable systems satisfy a hyperbolicity condition on the tangent map. A dominated splitting is a natural way to relax hyperbolicity.

THEOREM 3.2. *Let a, b, c, d be C^1-smooth immersions from S^1 to S^1 that preserve orientation. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be given by*

$$f(\theta, \phi) = (a(\theta) + b(\phi), c(\theta) + d(\phi)).$$

Then the positive cone field is preserved by Df and if f is a diffeomorphism then f has a dominated splitting on all of $\mathbb{T}^2$†.

Proof. Observe that

$$Df = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Let $\mathcal{C}^+$ be the positive cone bounded by the directions spanned by the vectors $(1, 0)$ and $(0, 1)$. Observe that $DF(\mathcal{C}^+)$ is properly contained in $\mathcal{C}^+$. The existence of a dominated splitting follows from standard arguments (see, for instance, [**BDV**, p. 293]). □

COROLLARY 3.3. *Given a two-dimensional Blaschke product diffeomorphism*

$$F(z, w) = (A(z)B(w), C(z)D(w)),$$

then $\hat{F}$ has a dominated splitting on all of $\mathbb{T}^2$ and the complexification of this dominated splitting is a dominated splitting for $F|\mathbb{T}^2$ with respect to the complex derivative of F.

THEOREM 3.4. *Let a, b, c, d be C^1-smooth immersions from S^1 to S^1 such that a, d preserve orientation and b, c reverse orientation. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be given by*

$$f(\theta, \phi) = (a(\theta) + b(\phi), c(\theta) + d(\phi)).$$

Then the positive cone field bounded by the directions spanned by the vectors $(1, 0)$ and $(0, -1)$ is preserved by Tf and if f is a diffeomorphism then f has a dominated splitting on all of $\mathbb{T}^2$‡.

Proof. The proof is similar to that of Theorem 3.2. □

COROLLARY 3.5. *Given a two-dimensional quotient Blaschke product diffeomorphism*

$$F(z, w) = \left(\frac{A(z)}{B(w)}, \frac{D(w)}{C(z)}\right),$$

then $\hat{F}$ has a dominated splitting on all of $\mathbb{T}^2$ and the complexification of this dominated splitting is a dominated splitting for $F|\mathbb{T}^2$ with respect to the complex derivative of F.

† If f is not a diffeomorphism there is a dominated splitting on the inverse limit space.
‡ If f is not a diffeomorphism there is a dominated splitting on the inverse limit space.

Now we can apply results from real dynamics about diffeomorphisms with dominated splittings (Theorem B in [**PS1**]). Recall that $\Omega(f)$ is the non-wandering set of f.

THEOREM 3.6. [**PS1**] *Let $f \in \mathrm{Diff}^2(M^2)$ and assume that $\Lambda \subset \Omega(f)$ is a compact invariant set exhibiting a dominated splitting such that any periodic point is a hyperbolic periodic point of saddle type. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugate to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

In the next two propositions we show that periodic simple closed curves do not occur for Blaschke product diffeomorphisms and that there is at most one sink.

PROPOSITION 3.7. *Let F be a Blaschke product diffeomorphism. Then F does not have any normally hyperbolic simple closed invariant curves.*

Proof. If C is an invariant simple closed curve for F then either C represents a non-trivial element of the fundamental group of $\mathbb{T}^2$ fixed by N_F or C bounds a disc. Since N_F is a hyperbolic linear map it has no non-trivial fixed elements, so the first eventuality is impossible. In the second case the bundle E which is tangent to C cannot be extended over the disc bounded by C contradicting the fact that there is a dominated splitting defined on all of $\mathbb{T}^2$. □

PROPOSITION 3.8. *A two-dimensional Blaschke product, F, has at most one attracting or semi-attracting periodic point, which is necessarily a fixed point, in $\mathbb{D} \times \mathbb{D}$.*

Proof. Since $\mathbb{D} \times \mathbb{D}$ is invariant for F, F^n is a normal family in $\mathbb{D} \times \mathbb{D}$. If $z_0 \in \mathbb{D} \times \mathbb{D}$ is an attracting or a semi-attracting periodic point of F then some subsequence F^{n_k} converges to the constant map z_0 on an open set in, and hence all of the interior of, $\mathbb{D} \times \mathbb{D}$. Let j be the period of z_0. It follows that $F^{jn}(z)$ converges to z_0 for all z in the interior of $\mathbb{D} \times \mathbb{D}$. Thus z_0 is the unique attracting periodic point in $\mathbb{D} \times \mathbb{D}$. □

QUESTION 3.9. *Is there a version of Proposition 3.8 which counts the number of repellors, even for Blaschke products which are birational equivalences?*

We can now describe the dynamics of Blaschke product diffeomorphisms.

THEOREM 3.10. *Let F be either a Blaschke product diffeomorphism or a quotient Blaschke product diffeomorphism such that any periodic point in $\mathbb{T}^2$ is a hyperbolic periodic point. Then, $F_{|\mathbb{T}^2}$ is an Axiom A diffeomorphism. Moreover, one of the following options holds:*

(1) $F_{|\mathbb{T}^2}$ *is Anosov and* $L(F_{|\mathbb{T}^2}) = \mathbb{T}^2$,

(2) $L(F_{|\mathbb{T}^2}) = \mathcal{S} \cup \mathcal{H} \cup \mathcal{S}a \cup \mathcal{R}$,

(3) $L(F_{|\mathbb{T}^2}) = \mathcal{S} \cup \mathcal{H}$,

(4) $L(F_{|\mathbb{T}^2}) = \mathcal{H} \cup \mathcal{S}a \cup \mathcal{R}$,

where $\mathcal{S}$ is a set formed by a single attracting fixed point, $\mathcal{R}$ is a set formed by a finite number of repelling periodic points, $\mathcal{S}a$ is a finite number of isolated saddles and $\mathcal{H}$ is a non-trivial maximal transitive hyperbolic invariant set in $\mathbb{T}^2$. In the last case it follows *that $\mathcal{H}$ is an attractor in $\mathbb{T}^2$. Moreover, the order relation is given by $\mathcal{R} \to \mathcal{S}a \to \mathcal{H} \to \mathcal{S}$ (where $A \to B$ if $W^u(A) \cap W^s(B) \neq \emptyset$). In the case when $\mathcal{S}$ is empty, $F|\mathbb{T}^2$ has a unique SRB measure with positive entropy.*

The proof of the theorem is long and complicated.

Proof. Step (1). Since F has a dominated splitting on all of $\mathbb{T}^2$, the complement of the basins of the periodic attracting and repelling points of F satisfies the hypotheses of Theorem 3.6. It is a hyperbolic set by Theorem 3.6 and Proposition 3.7. Arguing as in [**PS1**, §2] we see that F has only finitely many periodic attracting and repelling periodic points and is Axiom A. By Proposition 3.8, there is at most one attracting periodic point (which must be fixed and which we denote by $\mathcal{S}$). We denote the repelling periodic points by $\mathcal{R}$. If $\mathcal{S}$ and $\mathcal{R}$ are both empty then all of $\mathbb{T}^2$ is a hyperbolic set, and we are in Option (1).

Step (2). To see that one of our four options hold, it is sufficient to see that, with the exception of finitely many saddles (which we denote by $\mathcal{S}_a$), all the remaining hyperbolic saddles belong to one non-trivial homoclinic class (which we denote by $\mathcal{H}$), i.e. their stable and unstable manifolds intersect transversally. We prove this in Step (3). Transversality of the intersection of stable and unstable manifolds of saddles is guaranteed by dominated splitting, since unstable manifolds of saddles are tangent to $\mathcal{F}$ while stable manifolds are tangent to $\mathcal{E}$.

Step (3). From the fact that N_F is a hyperbolic linear map with positive entries it follows that $Trace(F_*^n) \to +\infty$ as $n \to \infty$ and this implies, by the Lefschetz formula and [**SS**] or Nielsen theory, that the number of periodic points goes to infinity. So $Closure(Per(F))$ contains a non-trivial homoclinic class. It remains to prove that there is a *unique* non-trivial homoclinic class. First, note that if p and q are distinct hyperbolic saddles and $W^s(p) \cap W^u(q) \neq \emptyset$ then $W^s(p)$ and $W^u(q)$ have infinite length. It follows from the finiteness of the spectral decomposition and local product structure for Axiom A basic sets that there are at most finitely many hyperbolic saddles with the property that their stable or unstable manifolds have finite length. We denote this set of saddles by $\mathcal{S}_a$. We now prove that the closure of the set of hyperbolic saddles with the property that at least one branch of its unstable manifold and at least one branch of its stable manifold is of unbounded length is a unique homoclinic class (denoted with $\mathcal{H}$). This will finish Step (3). For the proof we prove some lemmas.

LEMMA 3.11. *There exists $h : \mathbb{T}^2 \to \mathbb{T}^2$ continuous, onto and homotopic to the identity such that $h \circ F = A \circ h$, where A is the linear Anosov map induced by N_F. That is, F is semiconjugate to A. Moreover, let p be a periodic point of F such that one of its unstable branches has unbounded length. Then it follows that $h(\gamma^u(p)) \subset W^u(h(p))$ and $h(\gamma^u(p))$ has unbounded length, where $\gamma^u(p)$ is the branch of $W^u(p) \setminus \{p\}$ such that the corresponding unstable branch has unbounded length. The same holds for the stable branches.*

To prove Lemma 3.11 we use the following lemma.

LEMMA 3.12. *Let $\hat{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of F to $\mathbb{R}^2$. Then $\hat{F} = A + P$ such that A is a linear Anosov map and P is periodic and there exists $H : \mathbb{R}^2 \to \mathbb{R}^2$ continuous and* *onto such that* $H \circ \hat{F} = A \circ H$, *that is,* $\hat{F}$ *is semiconjugate to* A. *Moreover, there exists a constant* K_1 *such that* $\|H - \mathrm{Id}\| < K_1$. *Also, for any* $m \in \mathbb{Z}^2$ *we have* $H(x+m) = H(x) + m$.

Proof. The proof can be found in Theorem 2.2 of [**Fr**]. The main ingredient to construct the semiconjugacy is the global product structure of A. An alternate approach uses the fact that the shadowing lemma holds for any pseudo-orbit. □

Now we turn to the proof of Lemma 3.11.

Proof. Observe that Lemma 3.12 implies the existence of h. Moreover, since h is a semiconjugacy we also conclude that $h(\gamma^u(p)) \subset W^u(h(p))$. In fact, if $x \in W^u(p)$ then there exists $x_i \to p$ such that $f^i(x_i) = x$. So $h(x_i) \to h(p)$, $A^i(h(x_i) = h(f^i(x_i) = h(x)$ and $h(x) \in W^u(h(p))$. Similarly, semiconjugacies map stable manifolds to stable manifolds. We have to show now that the length of $h(\gamma^u(p))$ is unbounded. Since γ^u has infinite length we may find periodic saddle points whose stable manifolds cross γ^u in infinitely many points. Consider an embedded loop γ consisting of a connected subarc of γ^u and a short piece of stable arc β of a periodic saddle r. This loop is not contractable to a point. In fact, if it is not the case, let D be the disc bounded by γ. Observe that there it is a non-singular vector field X defined on D induced by the subbundle $\mathcal{F}$ of the dominated splitting with the property that X is inward pointing on D, which is a contradiction. Therefore, it follows that $N_F^n([\gamma]) = [F^n(\gamma)]$ (where N_F is the linear map acting on the fundamental group) goes to infinity. Since h induces the identity map on the fundamental group of $\mathbb{T}^2$, $[A^n(h(\gamma))]$ also goes to infinity in the fundamental group of $\mathbb{T}^2$. Since the length of $A^n(h(\beta))$ is bounded, the length of $A^n(h(\gamma^u))$ goes to infinity. □

In the next lemma we will use Lemma 3.11 to conclude that the closure of all the periodic points such that at least one branch of its unstable manifold and at least one branch of its stable manifold is not bounded is a unique homoclinic class.

LEMMA 3.13. *Let* p *and* q *be hyperbolic saddle points of* F *such that* p *has an unbounded unstable branch and* q *has an unbounded stable branch. Then these branches have non-empty intersection.*

Proof. Let p be a periodic point with an unbounded unstable branch and let q be a periodic point with an unbounded stable branch. Let $\gamma^u(p)$ be the branch of $W^u(p) \setminus \{p\}$ with unbounded length. Let $\gamma^s(q)$ be the branch of $W^s(q)) \setminus \{q\}$ with unbounded length. From Lemma 3.11 it follows that $h(\gamma^u(p))$ is an arc of unbounded length contained in the unstable manifold of $h(p)$. Since $W^u(h(p))$ and $W^s(h(q))$ intersect transversally (recall that, since A is a linear Anosov, for any pair of periodic point of A, any of their stable and unstable branches intersect each other), it follows that $h(\gamma^u(p))$ and $h(\gamma^s(q))$ intersect infinitely often and the long arcs in $h(\gamma^u(p))$ and $h(\gamma^s(q))$ have arbitrarily large intersections. Now if we complete a long arc in $\gamma^u(p)$ to a closed loop $\gamma(p)$ adding a small arc in $W^s(r_1)$ and a long arc in $\gamma^s(q)$ to a closed loop $\gamma(q)$ adding a small arc in $\gamma^u(r_2)$ for appropriately chosen saddles r_1, r_2, we see that $h(\gamma(p))$ and $h(\gamma(q)$ intersect homologically with as large an intersection number as we wish. Since h induces the identity map in homology so do $\gamma(p)$ and $\gamma(q)$, and then by construction so do $\gamma^u(p)$ and $\gamma^s(q)$. □

This concludes the proof of Step (4).

Step (5). If $\mathcal{S}$ is empty then $\mathcal{H}$ is an expanding attractor, which always carries a SRB measure of positive entropy. □

Using the spectral decomposition theorem for dominated splitting (see [**PS2**]) similar results as that stated in Theorem 3.10 can be obtained without assuming that the periodic points are hyperbolic.

4. *Families of Blaschke products*

Now, given a two-dimensional Blaschke product diffeomorphism $F(z, w) = (A(z)B(w), C(z)D(w))$, we consider the following two-parameter family:

$$F_{(\theta,\phi)}(z, w) = (\theta A(z)B(w), \phi C(z)D(w)), \quad (\theta, \phi) \in \mathbb{T} \times \mathbb{T}.$$

THEOREM 4.1. *Let F be a Blaschke product diffeomorphism. Then there exists an open set $U \subset \mathbb{T}^2$ such that if $(\theta, \phi) \in U$ then it follows that $F_{(\theta,\phi)|\mathbb{T}^2}$ satisfies the first or last option of Theorem 3.10, i.e. $F_{(\theta,\phi)}$ is Anosov or $L(F_{(\theta,\phi)|\mathbb{T}^2})$ is the union of a finite number of repelling periodic points and saddles and a hyperbolic non-trivial attractor. In either case $F_{(\theta,\phi)|\mathbb{T}^2}$ has a unique SRB measure of positive entropy.*

QUESTION 4.2. *Are there always parameter values (θ, ϕ) such that $F_{(\theta,\phi)}$ is Anosov?*

Proof. Let U be the set of parameter (θ, ϕ) such that $F_{(\theta,\phi)}$ has neither a sink nor a semi-attracting fixed point in $\mathbb{T}^2$. U is obviously open. We will show it is non-empty. Assume by contradiction that U is empty. Since, for any (θ, ϕ), $F_{(\theta,\phi)}$ has at most one attracting or a semi-attracting fixed point, then there is a continuous function from $\mathbb{T}^2$ to $\mathbb{T}^2$, $(\theta, \phi) \to s(\theta, \phi)$, such that $s(\theta, \phi)$ is either a sink or a semi-attracting fixed point for $F_{(\theta,\phi)}$. However, if $s(\theta, \phi)$ is a semi-attracting fixed point for $F_{(\theta,\phi)}$ for some (θ, ϕ), from the fact that $F_{(\theta,\phi)}$ has a dominated splitting, it follows that $s(\theta, \phi)$ is a saddle node and there is an invariant center manifold containing the saddle node; we can then choose a nearby parameter value to bifurcate the point to make it disappear. The resulting system then has no attracting or semi-attracting fixed point, contradicting the hypothesis that U is empty. Therefore, we may assume that $s(\theta, \phi)$ is a hyperbolic sink, with a dominated splitting for every value of (θ, ϕ). It follows that $\mathrm{Id} - DF_{(\theta,\phi)}(s(\theta, \phi))$ is an isomorphism and $|\det(\mathrm{Id} - DF_{(\theta,\phi)}(s(\theta, \phi)))| < 1$. By the implicit function theorem, $Ds(\theta, \phi) = (\mathrm{Id} - DF_{(\theta,\phi)}(s(\theta, \phi)))^{-1}$. So s is a covering map from the torus to itself and $|\det(Ds(\theta, \phi))| > 1$ for all (θ, ϕ). Hence the map s is at least two to one. However this is impossible since $F_{(\theta_1,\phi_1)}(x) = F_{(\theta_2,\phi_2)}(x)$ for any x in the torus implies $(\theta_1, \phi_1) = (\theta_2, \phi_2)$. So U is not empty.

To finish, it is enough to show that there is an open and dense set of parameter values such that all the periodic point are hyperbolic. By Theorem 3.10 it is enough to show there is a dense set of parameter values such that all the periodic points are hyperbolic since it follows from Theorem 3.10 that these systems are stable. This last fact follows from a standard Kupka–Smale argument. There are finitely many periodic points of any period, and if q is a non-hyperbolic periodic point, since F has a dominated splitting, we find an invariant center manifold containing the point q and we can choose a nearby parameter value to bifurcate the point into a hyperbolic one. It follows that the set of parameter values with diffeomorphisms with all periodic points of a given period hyperbolic is open and dense. Hence the set with all periodic points hyperbolic is residual and hence dense. □

Observe that in the case when $F_{(\theta,\phi)|\mathbb{T}^2}$ has a unique SRB measure $\mu_{(\theta,\phi)}$ of positive entropy, it follows that

$$h_{\mu_{(\theta,\phi)}} = \int_{\mathbb{T}^2} \log \|DF\| \, d\mu_{(\theta,\phi)} = \lambda^+_{(\theta,\phi)} > 0,$$

where $\lambda^+_{(\theta,\phi)}$ is the positive Lyapunov exponent. On the other hand, when $F_{(\theta,\phi)|\mathbb{T}^2}$ has either a sink or a semi-attracting fixed point, then the SRB measure is a Dirac measure on the sink or the semi-attracting fixed point, and therefore we can also define the metrical entropy related to these measures. We wonder whether it is possible to obtain lower bounds for

$$\int_{\mathbb{T}\times\mathbb{T}} h_{\mu_{(\theta,\phi)}} \, d\theta \, d\phi,$$

similar to those we have obtained in [**PRS**].

5. *Final remarks, questions and some generalizations*

5.1. *Julia sets.* Let $F : \mathbb{C}^2 \setminus P \to \mathbb{C}^2$ be an analytic function, where P is a set of poles given by a finite union of codimension-one submanifolds. Let us assume that, given F analytic on $\mathbb{C}^2 \setminus \mathcal{P}$, there exists a finite number of one-dimensional complex submanifolds $\{l_i\}$ such that F has an analytic inverse

$$F^{-1} : \mathbb{C}^2 \Big\backslash \Big[\bigcup_i l_i\Big] \to \mathbb{C}^2.$$

Observe that this is the case for the quotient Blaschke diffeomorphisms. It is possible to define $K^\pm$ as the sets

$$K^+ = \{(z, w) \in \mathbb{C}^2 \setminus P : \exists \text{ neighborhood } U \text{ of } (z, w) \text{ such that } F^n_{|U} \text{ is a normal family}\}$$

$$K^- = \Big\{(z, w) \in \mathbb{C}^2 \Big\backslash \Big[\bigcup_i l_i\Big] : \exists \text{ neighborhood } U \text{ of } (z, w) \text{ such that } F^{-n}_{|U} \text{ is a normal family and } \Big[\bigcup_i l_i\Big] \cap F^{-n}(U) = \emptyset \; \forall n > 0\Big\}.$$

We define the Julia set J as

$$J = \mathbb{C}^2 \Big\backslash \Big[K^+ \cup K^- \cup \bigcup_i l_i \cup P\Big].$$

It follows immediately that

$$\mathcal{H} \subset J(F).$$

QUESTION 5.1. *Is it true that*

$$\mathcal{H} = J(F)?$$

5.2. *Some general version for analytic diffeomorphisms.* Some of the result stated here can be stated in a more general setting using classical results of complex analysis (see [**R**]).

Remark 5.2. Any analytic function on $\mathbb{D} \times \mathbb{D}$ that keeps $\mathbb{T}^2$ invariant is a Blaschke product.

In fact, this can be easily checked in the following way. Fixing $w \in S^1$, it follows that $z \to F_1(z, w)$ (where F_1 is the first coordinate of F) is an analytic function on $\mathbb{D}^1$ that keeps S^1 invariant and, therefore, is a one-dimensional Blaschke product

$$F_1(z, w) = B(w) \prod_{i=1}^{n} \frac{z - a_i(w)}{1 - z\overline{a_i(w)}},$$

where $w \to B(w)$ is an analytic function that preserves S^1 and $\prod_{i=1}^{n} (z - a_i(w))/(1 - z\overline{a_i(w)})$ is an analytic function. In particular, it follows that $B(w)$ is also a Blaschke product. To finish, observe that in the quotient complex conjugation is used so, to keep analyticity, it follows that the symmetric functions of the $a_i(w)$ are constant functions.

Using the Riemann mapping theorem and Theorem 3.10 the next remark follows.

Remark 5.3. Let F be an analytic map in $cl(U) \times cl(V) \to \mathbb{C}^2$ such that U, V are two connected open sets in $\mathbb{C}$. Let us assume that $F(U \times V) \subset U \times V$, $F(\partial U \times \partial V) = \partial U \times \partial V$, and $F_{|\partial U \times \partial V}$ is a diffeomorphism (where ∂W denotes the boundary of W). Then, the thesis of Theorem 3.10 holds on $\partial U \times \partial V$.

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