DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **14**, Number **4**, April **2006**

Website: http://aimSciences.org

pp. **845–855**

UNIQUE ERGODICITY, STABLE ERGODICITY, AND THE MAUTNER PHENOMENON FOR DIFFEOMORPHISMS

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(Communicated by Marcelo Viana)

Abstract. In 1954, F. Mautner gave a simple representation theoretic argument that for compact surfaces of constant negative curvature, invariance of a function along the geodesic flow implies invariance along the horocycle flows (these are facts which imply ergodicity of the geodesic flow itself), [M]. Many generalizations of this Mautner phenomenon exist in representation theory, [St1]. Here, we establish a new generalization, Theorem 2.1, whose novelty is mostly its method of proof, namely the Anosov-Hopf ergodicity argument from dynamical systems. Using some structural properties of Lie groups, we also show that stable ergodicity is equivalent to the unique ergodicity of the strong stable manifold foliations in the context of affine diffeomorphisms.

1. **Introduction.** Beginning with [GPS] the first two authors have been studying stable ergodicity of volume preserving partially hyperbolic diffeomorphisms on a compact manifold M. The most recent survey on the subject is [PS3]. A key issue is the way in which the strong stable and strong unstable manifolds foliate M. To prove ergodicity one assumes essential accessibility, namely that every Borel set $S \subset M$ which consists simultaneously of whole strong stable leaves and whole strong unstable leaves has measure zero or one. Such a set S is said to be us-saturated. As essential accessibility is a measure theory concept, it is difficult to verify and even more difficult to prove stable under perturbation. A stronger assumption is full accessibility[1] in which it is required that M and the empty set are the only us-saturated sets. In many cases full accessibility is stable under perturbation, and this leads to stable ergodicity.

The second author was supported in part by an NSERC Discovery Grant.
The third author was supported by the Leading Scientific School Grant, No 457.2003.1.

[1] In previous papers, we referred to full accessibility as us-accessibility, and to a stronger condition as homotopy accessibility. The latter is always stable under perturbation and is often a consequence of the former.

We have conjectured that the stably ergodic diffeomorphisms are open and dense among C^2 volume preserving partially hyperbolic diffeomorphisms. Our plan of attack was to prove that an open and dense subset of partially hyperbolic diffeomorphisms are fully accessible. As already noted, this seems far easier than the similar assertion for essential accesssibility, so we focused on the full accessibility property. The following recent developments, however, caused us to reconsider our position and to shift our attention more in the direction of essential accessibility.

(a) Among affine diffeomorphisms of finite volume, compact homogeneous spaces, those which are stably ergodic among left translations are precisely those with the essential accessibility property [St3]. (These are also precisely those which are K-automorphisms.) In other words, affine stable ergodicity is equivalent to essential accessibility. The proof relies significantly on the structural properties of Lie groups.

(b) As was shown by Federico Rodgriguez Hertz, essential accessibility without full accessibility sometimes leads to (nonlinear) stable ergodicity, [RH].

(c) The Mautner phenomenon from representation theory leads to a proof of half of the affine stable ergodicity result mentioned in (a), while in [PS2] we establish a nonlinear version of the Mautner phenomenon, which we apply to nonlinear stable ergodicity.

Below, we give a proof of the Mautner phenomenon in the case it is used for (a), but instead of structural properties of Lie groups or their representation theory, we use Birkhoff's theorem as in the Hopf-Anosov argument for the ergodicity of Anosov systems. This proof makes us feel we have landed in the right place.

We would like to see a unified explanation of these Mautner phenomena, one that might generalize Rodriguez-Hertz's theorem to *all* essentially accessible affine diffeomorphisms. To this end we wondered what more of a potentially useful nature could be said about the strong stable and unstable manifold foliations. The third author has extended the results in his monograph [St1], and answered question 6.8 of [BPSW] for affine diffeomorphisms – namely, in the affine, essentially accessible case, the strong stable or strong unstable manifold foliations are uniquely ergodic. See Theorem 3.1 below.

2. **The Mautner phenomenon.**

Roughly speaking the Mautner phenomenon refers to invariance of a function along trajectories of one flow implying invariance along certain transverse flows. Mautner first observed the phenomenon in an affine ergodicity proof – invariance of a function along the geodesic flow (for a compact surface of constant negative curvature) implies invariance along the horocycle flows. See [M]. It has been generalized considerably by Auslander-Green, Dani and Moore for the ergodic theory of flows on homogeneous spaces. See [St1] for references, proofs and a discussion of the results.

A version of the Mautner phenomenon applies precisely to prove the ergodicity of the essentially accessible affine diffeomorphisms of finite volume, compact homogeneous spaces. In [PS3] we sketched a proof of this result. Below, we do a better job. The proof is quite close in structure to the best proof we have for partially hyperbolic diffeomorphisms with the essential accessibility property, see [PS2] and [PS3].

Let G be a connected Lie group, and $B \subset G$ a closed subgroup such that G/B is compact and of finite volume, i.e., G/B admits finite G-invariant volume.[2] Let $f \in \mathrm{Aff}(G/B)$ be an affine map of G/B, i.e., $f = L_a \circ A$, where $L_a : G/B \to G/B$ is left translation by a fixed element $a \in G$ and $A : G/B \to G/B$ is a map induced by a fixed automorphism $\overline{A} \in \mathrm{Aut}(G)$ such that $\overline{A}(B) = B$. The covering map $\overline{f} = L_a \circ \overline{A} : G \to G$ makes

$$\begin{tikzcd} G \arrow[r, "\overline{A}"] \arrow[d] & G \arrow[r, "L_a"] \arrow[d] & G \arrow[d] \\ G/B \arrow[r, "A"] & G/B \arrow[r, "L_a"] & G/B \end{tikzcd}$$

commute and induces an automorphism $d\overline{f}$ of the Lie algebra $\mathfrak{g}$. With respect to $d\overline{f}$, the Lie algebra $\mathfrak{g}$ splits into generalized eigenspaces $\mathfrak{g} = \mathfrak{g}^u \oplus \mathfrak{g}^c \oplus \mathfrak{g}^s$ such that the eigenvalues of $d\overline{f}$ are respectively outside, on, or inside the unit circle. The corresponding connected subgroups G^u, G^c, and G^s are the unstable, central, and stable horospherical subgroups. Their orbits form the (strong) unstable, center, and (strong) stable foliations for f on G/B. These facts are proved in [PSS].

Let $H \subset G$ be the subgroup generated by G^u and G^s. It is normal and called the hyperbolic subgroup for f. See [PS1]. Under the previous conditions, the version of the Mautner phenomenon that we prove is:

Theorem 2.1. *Every f-invariant L^1 function $\phi : G/B \to \mathbb{R}$ is essentially constant on cosets $x\overline{HB}$.*

Since the values of an L^1 function are ambiguous on a zero set, the meaning of Theorem 2.1 is this: we assume that for almost every $y = xB \in G/B = M$, $\phi(f(y)) = \phi(y)$, and we conclude that there is a zero set $Z \subset M$ such that for each pair $y, y' \in M \setminus Z$ that lie in a common left coset of $\overline{HB}$, we have $\phi(y) = \phi(y')$.

Corollary 2.2. *If $\overline{HB} = G$ then f is ergodic.*

Remark. According to Dani (see [St1] §1), f is a K-automorphism if and only if $G = \overline{HB}$. $G = \overline{HB}$ is the same as essential accessibility and equivalent to the stable ergodicity of f among left translation perturbations [St3].

We give the proof of the theorem modulo some facts which should be standard and the Hopf-Anosov argument which are proven below.

Proof of Theorem 2.1. We are given an affine manifold $M = G/B$ and an L^1 function $\phi : M \to \mathbb{R}$ which is invariant under the affine diffeomorphism $f : M \to M$, and we claim that ϕ is essentially constant along left cosets of $\overline{HB}$. By the Hopf-Anosov argument (Lemma 2.4), ϕ is essentially constant along the leaves of the strong stable and strong unstable foliations, which, as stated above, are the left orbits of Lie subgroups G^s and G^u. That ϕ is invariant along the left orbits of G^s and G^u is equivalent to the fact that ϕ is g invariant for every g in G^s and G^u

[2] "Finite volume" includes the requirement that the measure on G/B be G-invariant. In particular, if $G = \mathrm{SL}(2, \mathbb{R})$, Γ is a uniform discrete subgroup, and T is the subgroup of upper triangular matrices then

(a) the homogeneous space G/Γ is of finite volume, but
(b) the homogeneous space $G/T \approx S^1$ is not of finite volume because there is no G-invariant measure on it.

(Lemma 2.7), that is there is a zero set Z such that for all $g \in G^u \cup G^s$ and for all $y, gy \in M \setminus Z$ we have $\phi(y) = \phi(gy)$. We claim that ϕ is also essentially constant along H-orbits, where H is the subgroup generated by G^u and G^s.

Because H is normal, its orbits foliate M. Take any $h \in H$. It is expressed as a product

$$h = g_1 \cdot g_2 \cdots g_n$$

where $g_1, \dots, g_n$ are alternately in G^u and G^s. Now

$$Z_h = Z \,\cup\, g_1^{-1}Z \,\cup\, \dots \,\cup\, g_n^{-1}Z$$

is a zero set, and for all $y \in M \setminus Z_h$ we have $\phi(y) = \phi(hy)$. Using Lemma 2.7 again implies that ϕ is essentially constant along left H-orbits. Since H is normal, left orbits are the same as right orbits and Lemma 2.7 gives a zero set Z^r such that for all $h \in H$ and for all $y, yh \in M \setminus Z^r$ we have

$$\phi(y) = \phi(yh)$$

where $yh = xhB$ and $y = xB$. Clearly we also have

$$\phi(yhb) = \phi(xhbB) = \phi(xhB) = \phi(yh) = \phi(y),$$

so HB is contained in the stabilizer of ϕ. Now the stabilizer of ϕ is closed (Lemma 2.8) so the stabilizer contains also the closure $\overline{HB}$. Thus, for each $g \in \overline{HB}$ we have a zero set Z_g such that for all $y \in M \setminus Z_g$, $\phi(y) = \phi(yg)$. One more application of Lemma 2.7 implies that ϕ is essentially constant along left cosets of $\overline{HB}$. $\square$

Now we turn to the lemmas used in the proof of Theorem 2.1. We make several applications of Fubini's Theorem, all of which the well educated reader may find superfluous.

We say that ϕ is essentially constant along the cells of a partition $\mathcal{P}$ of M if, excluding a zero set from M, $\phi(x) = \phi(x')$ whenever x, x' lie in a common cell of $\mathcal{P}$.

Lemma 2.3. *Suppose that X, Y are σ-finite measure spaces and $\phi_n : X \times Y \to \mathbb{R}$, $n = 1, 2, \dots,$ is a sequence of measurable functions that converges almost everywhere to a limit ϕ. If each ϕ_n is essentially constant along the slices $X \times y$ then the same is true for ϕ.*

Proof. The hypothesis means that for each n, there is a zero set $Z_n \subset X \times Y$, and if we call its y-slice

$$Z_n(y) = Z_n \cap (X \times y)$$

then the function $\phi_n(\,\cdot\,, y)$ is constant on $(X \times y) \setminus Z_n(y)$. By Fubini's Theorem the set of slices $X \times y$ for which $Z_n(y)$ has positive X-measure is a zero set Z'_n. Let Z be the union of the zero sets Z'_n and the zero set on which $\phi_n(x, y)$ fails to converge to $\phi(x, y)$. It is a zero set in $X \times Y$. On each y-slice not in Z, $\phi_n(x, y)$ is almost everywhere constant and converges pointwise to $\phi(x, y)$. Hence $\phi(x, y)$ is essentially constant along the y-slices. $\square$

The next lemma contains the application of Birkhoff's theorem in the Hopf-Anosov proof of ergodicity. We have stated and proved it more generally than it is employed here as the foliations in our application are smooth and hence absolutely continuous.

Lemma 2.4. *If $f : M \to M$ is a C^2 measure preserving, partially hyperbolic diffeomorphism and $\phi : M \to \mathbb{R}$ is a measurable f-invariant function then ϕ is essentially constant along the leaves of the strong stable and strong unstable foliations.*

Proof. Suppose at first that ϕ is L^1. The Birkhoff Ergodic Theorem provides a continuous projection $\beta : L^1(M, \mathbb{R}) \to \mathrm{Inv}^1(f)$ where

$$\beta(\psi)(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(x)),$$

the limit exists almost everywhere, and $\mathrm{Inv}^1(f)$ is the space of L^1 invariant functions. Clearly $\beta(\phi) = \phi$. If ψ is continuous then it is straightforward to see that if $\beta(\psi)(x)$ exists at one point of a strong stable manifold $W^{ss}(p)$ then it exists at all points of $W^{ss}(p)$ and has the same value. Thus, $\beta(\psi)$ is essentially constant along the leaves of the strong stable foliation.

The space $C^0(M, \mathbb{R})$ is dense in $L^1(M, \mathbb{R})$, and so there is a sequence of continuous functions ψ_n that converges to ϕ in the L^1 sense. Since β is continuous, $\beta(\psi_n)$ converges to ϕ in the L^1 sense. The Riesz Lemma gives a subsequence such that

$$\lim_{k\to\infty} \beta(\psi_{n_k})(x) = \phi(x) \text{ almost everywhere.}$$

Applying Lemma 2.3 in a $\mathcal{W}^{ss}$-foliation box is permissible because the foliation is absolutely continuous. Hence ϕ is essentially constant along strong stable plaques. Covering M with foliation boxes completes the proof that ϕ is essentially constant along strong stable leaves. Replacing f with f^{-1} gives the same assertion for the strong unstable foliation.

Finally, if ϕ is measurable but not L^1, we replace it by a cut-off version

$$\phi_L(x) = \begin{cases} \phi(x) & \text{if } |\phi(x)| \le L \\ 0 & \text{if } |\phi(x)| > L. \end{cases}$$

Clearly, ϕ_L is L^1 and f-invariant. Essential constancy of ϕ_L along the leaves of the strong stable and strong unstable foliations implies the same for ϕ. □

Lemma 2.5. *If a measurable function ϕ is almost everywhere constant on almost every leaf of an absolutely continuous foliation $\mathcal{F}$ then it is essentially constant along the leaves of $\mathcal{F}$.*

Proof. Let $\phi : M \to \mathbb{R}$ be such a function. By hypothesis there is a zero set $Z_0 \subset M$ that consists of whole $\mathcal{F}$-leaves, and for each $\mathcal{F}$-leaf $L \subset M \setminus Z_0$ there are a constant $c(L)$ and a leaf-zero-set $Z_L \subset L$ such that $f(z) = c(L)$ for all $z \in L \setminus Z_L$. Define

$$\psi(z) = \begin{cases} c(L) & \text{if } z \in L \subset M \setminus Z_L \\ \infty & \text{if } z \in Z_0 \end{cases}$$

It is enough to show that ψ is measurable. For then

$$Z = \{z \in M : \phi(z) \ne \psi(z)\}$$

is measurable and meets each leaf in a zero set. Since Z is a measurable set which is the union of leaf zero sets, it is a zero set. (Absolute continuity of $\mathcal{F}$ is needed to apply Fubini's Theorem, while measurability of Z is needed to infer that Z is a zero set. For recall that there is a nonmeasurable set in the plane which meets every horizontal line in a single point.) Thus if $x, x' \in M \setminus Z$ lie in a common leaf then

$\phi(x) = \psi(x) = \psi(x') = \phi(x')$, which means that ϕ is essentially constant along the leaves of $\mathcal{F}$.

To check measurability of ψ, cover the manifold by foliation boxes $U = X \times Y$, inside which the leaf-plaques are discs $X \times y$. Disintegrate the smooth measure m on M to plaque measures m_y. If S is measurable and $S_y = S \cap (X \times y)$ then for almost every y, S_y is plaque measurable, $y \mapsto m_y(S_y)$ is a measurable function, and

$$m(S \cap U) = \int m_y(S_y)\, dy.$$

Fix any $c \in \mathbb{R}$ and apply this to the sublevel set $S = \phi^{-1}(-\infty, c]$. The set is measurable and if $(x, y) \in U$ then

$$\psi(x, y) \leq c \iff m_y(S_y) > 0.$$

Measurability of $y \mapsto m_y(S_y)$ implies that $\{y : m_y(S_y) > 0\}$ is measurable and therefore

$$\{z \in U : \psi(z) \leq c\}$$

is measurable. A locally measurable set is measurable, so the sublevel set $\psi^{-1}(-\infty, c]$ is measurable, and ψ is measurable. $\square$

Remark. The converse to Lemma 2.5 is immediate.

Lemma 2.6. *If a diffeomorphism almost leaves a measurable set S invariant then it does leave its set of density points S^* invariant.*

Proof. We assume that $f : M \to M$ is a diffeomorphism, $Z = S \triangle fS$ is a zero set, and prove that $fS^* = S^*$. We have

$$S \setminus Z \subset fS \subset S \cup Z$$

and, since zero sets have no effect on density points,

$$S^* = (S \setminus Z)^* \subset (fS)^* \subset (S \cup Z)^* = S^*,$$

which gives $(fS)^* = S^*$. Since a diffeomorphism preserves every density point, $(fS)^* = fS^*$, which gives $fS^* = S^*$. $\square$

The next lemma generalizes the fact that almost everywhere invariance of a measurable function along orbits of a flow is implied by almost everywhere invariance for each time-t map.

Lemma 2.7. *Suppose that the smooth manifold M carries a smooth measure m, $\phi : M \to \mathbb{R}$ is a measurable function, G is a Lie group that acts smoothly on M, and the G-orbits foliate M. The following are equivalent:*

(a) *ϕ is essentially constant along the orbits of G.*
(b) *For each $g \in G$, there is a zero set $Z_g \subset M$ such that for all $x \in M \setminus Z_g$, $\phi(x) = \phi(gx)$.*

Proof. We write the action of G on M as $x \mapsto gx$.

Assume (a). Then there is a zero set $Z \subset M$ such that if $x, x' \in M \setminus Z$ belong to a common G-orbit then $\phi(x) = \phi(x')$. Fix $g \in G$ and define

$$Z_g = Z \cup g^{-1}Z.$$

Since G acts smoothly, Z_g is a zero set. If $x \in M \setminus Z_g$ then $x, gx \in M \setminus Z$, and by (a), $\phi(x) = \phi(gx)$. This gives (b).

Assume (b). We must find a zero set $Z \subset M$ such that for each $g \in G$, if $x, gx \in M \setminus Z$ then $\phi(x) = \phi(gx)$. Uncountability of G precludes taking Z to be the union of the Z_g, as g ranges through G. Since ϕ is measurable and the orbit foliation is smooth, there is at most a zero set of orbits Z_{NM}, restricted to which ϕ is nonmeasurable. Without loss of generality we assume that $Z_{\mathrm{NM}} = \emptyset$. Thus, each sublevel set of ϕ meets each orbit in an orbit measurable set.

Fix $a < b$ and set $A = \phi^{-1}(-\infty, a]$, $B = \phi^{-1}[b, \infty)$. The sets A, B are measurable, and meet each orbit in measurable sets. We claim that

$$Z(a,b) = \{z \in M : \text{both } A \text{ and } B \text{ meet the orbit } Gz \\ \text{in a set of positive orbit measure}\}$$

is a zero set.

Suppose that $Z(a,b)$ is not a zero set. Let A^* and B^* be the sets of density points of A and B. Since $A \triangle A^*$ is a zero set, Fubini's Theorem implies that the set of orbits meeting A with positive orbit measure differs by a zero set from the set of orbits meeting A^* with positive orbit measure. Thus, there are many orbits Gz such that A^* and B^* both meet Gz. Any such orbit gives $p \in A^*$ and $q = gp \in B^*$ for some $g \in G$.

By (b), there is a zero set Z_g such that if $x \in M \setminus Z_g$ then $\phi(gx) = \phi(x)$. Thus, if $x \in A \setminus Z_g$, we have $\phi(gx) = \phi(x) \le a$, which gives

$$g(A \setminus Z_g) \subset A.$$

Likewise, $g^{-1}(A \setminus Z_{g^{-1}}) \subset A$, so $A \setminus Z_{g^{-1}} \subset gA$, and

$$A \triangle gA \subset Z_{g^{-1}} \cup gZ_g,$$

which shows that A is almost g-invariant. By Lemma 2.6, $gA^* = A^*$. But then $gp = q \in A^*$, which contradicts the fact that disjoint sets have disjoint sets of density points. Therefore $Z(a,b)$ is a zero set.

Let Z_0 be the union of the orbits in $Z(a,b)$ as a, b range through pairs of rationals with $a < b$. It is a zero set. On each G-orbit not in Z_0, ϕ is almost everywhere constant. By Lemma 2.5, ϕ is essentially constant along the G-orbits, which is (a). □

Lemma 2.8. *If $\phi : M \to \mathbb{R}$ is measurable and $\alpha : G \to \mathrm{Homeo}(M)$ is a nice action then the stabilizer*

$$\mathrm{St}(\phi) = \{g \in G : \ \phi(x) = \phi(\alpha_g(x)) \ a.e.\}$$

is a closed subgroup of G.

Proof. Here, "nice" means that M is locally compact, metrizable, α is continuous, μ is a regular probability measure on M, the Radon-Nikodym derivatives of α_g exist and are locally uniformly bounded. In the case at hand, μ is a G-invariant measure on the homogeneous space $M = G/B$, and the action is left or right G-multiplication.

Suppose that $g, g' \in \mathrm{St}(\phi)$. Absolute continuity implies that $\alpha_{g'}$ is a zero-set-preserving change of variables. Hence

$$\phi \circ \alpha_g = \phi \ \ (a.e.) \Rightarrow \ \ \phi \circ \alpha_g \circ \alpha_{g'} = \phi \circ \alpha_{g'} = \phi \ \ (a.e).$$

Since $\alpha_{gg'} = \alpha_g \circ \alpha_{g'}$, we have $gg' \in \mathrm{St}(\phi)$. Similarly, absolute continuity of $\alpha_{g^{-1}}$ gives

$$\phi \circ \alpha_g = \phi \ \ (a.e.) \Rightarrow \ \ \phi \circ \alpha_g \circ \alpha_{g^{-1}} = \phi \circ \alpha_{g^{-1}} \ \ (a.e),$$

and hence $g^{-1} \in \mathrm{St}(\phi)$, which completes the proof that the stabilizer is a subgroup of G.

To prove closedness, suppose that $g_n \to g$ and $g_n \in \mathrm{St}(\phi)$ for all n. Call $\alpha_{g_n} = h_n$ and $\alpha_g = h$. We must show that $\phi \circ h = \phi$ almost everywhere. Since the issue is local, it is enough to choose a compact neighborhood N of an arbitrary $x_0 \in X$ and show that $\phi \circ h = \phi$ almost everywhere on N. Continuity of the action and compactness of N imply that $h_n|_N \to h|_N$ uniformly. Thus there is a compact neighborhood W of $h(N)$ such that for all $n \geq$ some n_0, we have

$$h_n(N) \subset W.$$

Lusin's Theorem states that ϕ is uniformly continuous on a compact subset $K \subset W$ where we can make $\mu(W \setminus K)$ as small as we want. Uniform local boundedness of the Radon-Nikodym derivatives implies that we can thereby force $\mu(N \setminus h^{-1}K)$ and $\mu(N \setminus h_n^{-1}K)$ to be as small as we want.

Let $\epsilon > 0$ be given. Choose K as above so that for each $n \geq n_0$,

$$\mu(S_n) < \epsilon \text{ where } S_n = N \setminus (h^{-1}(K) \cup h_n^{-1}(K)).$$

Metrize W with some metric d. There is a $\delta > 0$ such that if $y, y' \in K$ and $d(y, y') < \delta$ then $|\phi(y) - \phi(y')| < \epsilon$. There is also an $n_1 \geq n_0$ such that for each $n \geq n_1$ and each $x \in N$ we have

$$|h_n(x) - h(x)| < \delta.$$

Hence, for each $n \geq n_1$ and for all $x \in N \setminus S_n$ we have

$$|\phi \circ h_n(x) - \phi \circ h(x)| < \epsilon,$$

and consequently

$$\mu\{x \in N : |\phi \circ h_n(x) - \phi \circ h(x)| \geq \epsilon\} < \epsilon.$$

This means that $\phi \circ h_n|_N$ converges to $\phi \circ h|_N$ in measure. By Riesz's Lemma, there is a subsequence converging almost everywhere. Since $\phi \circ h_n = \phi$, we get $\phi \circ h|_N = \phi|_N$ almost everywhere, and hence $g \in \mathrm{St}(\phi)$ as claimed. □

3. **Unique Ergodicity.** In this section we discuss the higher ergodic properties of an affine diffeomorphism $f : M \to M$ under the following *standing hypotheses*:

(a) $M = G/B$ is compact, G is a connected Lie group, B is a closed subgroup, and M has a finite G-invariant volume.
(b) A is a fixed automorphism of G such that $AB = B$. We also denote the quotient map as $A : M \to M$.
(c) $f = L_a \circ A$ where a is a fixed element of G.

Our main theorem is

Theorem 3.1. *The following are equivalent:*

(1) *f is a K-automorphism of M.*
(2) *G^s is ergodic on M.*
(3) *G^s is minimal on M.*
(4) *G^s is strictly ergodic on M.*

According to Dani (see [St1] §1), f is a K-automorphism if and only if $G = \overline{HB}$, and we usually express the K-condition in this form. Recall that strict ergodicity is unique ergodicity plus minimality.

There are some known relations between these conditions, namely (4) $\Rightarrow$ (3) $\Rightarrow$ (2). Furthermore, the G^s-action on $G/\overline{HB}$ is trivial because $G^s \subset H$ and H is normal in G. Hence (2) $\Rightarrow$ (1). In all these relations M need not be compact but it should have finite volume.

It is known by the fundamental results of Ratner on unipotent flows (see [St1], Chapter II]) that (3) $\Rightarrow$ (4). To prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) we need the following result of Witte.

Theorem 3.2. [W] *Let $M = G/B$ be a finite volume homogeneous space, and let f be a weak mixing affine automorphism of M. Then*

(1) *The radical of G is nilpotent.*
(2) *B is Zariski dense in the adjoint representation* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. *In particular, any connected subgroup of G normalized by B, is normal in G.*

Let us prove that (1) $\Rightarrow$ (2). Note that G^s is a unipotent subgroup of G, and according to the Mautner phenomenon (see [St1], Section 2), the ergodic decomposition for the G^s-action on G/B is of the form $\{x\overline{NB},\ x \in G\}$, where $N \subset G$ is the smallest normal subgroup of G containing G^s. Since B is Zariski dense in the Ad-representation, it follows that $N' = (\overline{NB})_0$ is normal in G. Clearly, $\overline{A}(N) = N$. Since $\overline{A}(B) = B$, it follows that $\overline{A}(N') = N'$ and f covers an affine map $f' : M' \to M'$, where $M' = G'/N'$, $G' = G/N'$, and $B' = N'B/N'$. Since M' is of finite volume, it follows that f' keeps G'-invariant measure on M' invariant. But the stable horospherical subgroup for f' in G' is trivial, and the same is valid for the unstable horospherical subgroup. It follows that $G^u \subset N'$ and then $H \subset N'$. But since f is a K-automorphism, $G = \overline{HB}$. Hence $N' = G$, and thus the G^s-action on M is ergodic.

It remains to show that (2) $\Rightarrow$ (3), but we do not know how to do this directly. Instead, we derive it fairly naturally in the semisimple context from a classical result, Theorem 3.3. Then we discuss a few extensions beyond the semisimple case, and finally we prove the general theorem in a rather different way.

Recall that f is *semisimple* if $d\overline{f} : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable over $\mathbb{C}$.

Theorem 3.3. [B],[V],[EP] *If f is semisimple then the conditions (2), (3), and (4) from Theorem 3.1 are equivalent to f being weak mixing.*

Corollary 3.4. $(1) - (4)$ *are equivalent when f is semisimple or a pure translation.*

Proof. For such an f we claim that (1) $\Rightarrow$ (4).

Suppose that f is a semisimple. By (1) it is a K-automorphism, so it is weak mixing, and Theorem 3.3 gives (4).

Suppose that f is a pure translation, $f = L_a$ for some fixed $a \in G$. Since the radical of G is nilpotent, it follows that a admits a Jordan decomposition $a = s \times u$, where s is semisimple and u is unipotent. Since s and a have common horospherical subgroups, it follows $f = L_a$ and L_s share the same hyperbolic subgroup, and that condition (1) or (4) is valid for f if and only if it is valid for L_s. Thus (1) for $f = L_a$ implies (1) for L_s, and as we showed above, this implies (4) for L_s which implies (4) for f.

We have already observed that (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), so the conditions are equivalent. $\square$

Trying to extend these ideas to the general case, we observe that it is no loss of generality to assume that B is a *uniform lattice* in G, i.e., a discrete subgroup whose quotient space G/B is compact. In fact, let $D \subset B$ be the maximal connected subgroup that is normal in G. Since $\overline{A}(B) = B$, it follows that $\overline{A}(D) = D$. So we can replace G/B by G'/B', where $G' = G/D$ and $B' = B/D$. Hence we can assume that D is trivial, i.e., B is a *quasi-lattice* in G. On the other hand, by Theorem 3.2 its identity component B_0 is normal in G. So it is trivial, i.e., B is in fact discrete.

Now let $f = L_a \circ A$. One can try to reduce this to the pure translation case handled in Corollary 3.4 by using a suspension construction as in [PSS]. The idea is to embed some power A^k into one-parameter subgroup $C \subset \mathrm{Aut}(G)$ and replace $M = G/B$ by $M' = G'/B'$, where $G = C \cdot G'$ and $B' = A^{k\mathbb{Z}} \cdot B$. Then M' is invariant under pure translation L_h for some $h = h(a, A, k) \in G'$, and the action of L_h on M coincides with that of f^k. So, in a sense, the general case reduces to the pure translation case. The problem, however, is that L_h is only ergodic on M' and not even weak mixing. (Clearly, the G^s-action on M' also fails to be ergodic.)

So we need something else. One can try to find a semisimple affine map having the same horospherical subgroups as f as was done for pure translations. This is easy to do in the cases that G is either semisimple or nilpotent.

In fact, let G be semisimple. Then it is well known that the group of inner automorphisms is of finite index in $\mathrm{Aut}(G)$. Hence there exists $k \in \mathbb{N}$ and $g \in G$ such that for all $x \in G$, $A^k(x) = gxg^{-1}$. It follows that $f^k = L_h$ for some $h = h(a, A, k)$, and we fall into the pure translation case.

Next consider the abelian case $G = \mathbb{R}^n$, $B = \mathbb{Z}^n$ and assume for simplicity that $f = A \in \mathrm{SL}(n, \mathbb{Z})$. It is well known that for some $k \in \mathbb{N}$, $A^k = S \times U$, where S is semisimple, U is unipotent, and both S and U are in $\mathrm{SL}(n, \mathbb{Z})$. Hence in this case one can replace $f^k = A^k$ by its semisimple part S^k.

This idea generalizes to the nilpotent case: for some $k \in \mathbb{N}$, f^k admits Jordan decomposition inside $\mathrm{Aff}(G/B)$ into semisimple and unipotent parts. As for the general case, by Theorem 3.2, (1) implies that G is a semidirect product $G = P \cdot N$, where P is a semisimple subgroup and N is the nilradical of G. Apparently, combining semisimple and nilpotent cases, one can decompose some power of f into commuting semisimple and unipotent affine maps on G/B to apply Theorem 3.3.

However, this seems rather technically involved and in the general case we prefer not to use Theorem 3.3 directly.

To summarize, it remains to prove that (2) $\Rightarrow$ (3) for $f = L_a \circ A$; i.e., ergodicity of the G^s-action implies minimality. If G is semisimple, this can be deduced from Theorem 3.3 as above. Also, there is no problem if G is nilpotent. In fact, by Furstenberg's theorem (see [St1], §3) any ergodic homogeneous flow on a nilmanifold is minimal and strictly ergodic. In the general case this approach can be extended as follows.

Theorem 3.5. [St2] *Let f be an affine diffeomorphism on a compact homogeneous space $M = G/B$ of finite volume. Then the G^s-action on each invariant homogeneous subspace $x\overline{HB} \subset M$ is ergodic and minimal.*

Proof of Theorem 3.1. Assume (1). Then $\overline{HB} = G$. By Theorem 3.5, G^s is minimal on M, which is (3). As we have seen above, this gives equivalence of (1) – (4). □

Remark. The proof of Theorem 3.5 develops methods used by Dani in his study of horospherical flows (see [St1], Section 13) and eventually involves Theorem 3.3. The formulation and the proof were given in [St2] for pure translations only. However, using the suspension construction as above, one easily reduces to the pure translation case. Note that no assumptions on ergodic properties of the f-action are needed in Theorem 3.5.

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Received October 2004; revised August 2005.

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