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Partial Differentiability of Invariant Splittings*

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A key feature of a general nonlinear partially hyperbolic dynamical system is the absence of differentiability of its invariant splitting. In this paper, we show that often partial derivatives of the splitting exist and the splitting depends smoothly on the dynamical system itself.

KEY WORDS: Anosov diffeomorphisms; bundle splittings; partial hyperbolicity; invariant section theorem; bunching conditions; weak continuity.

1. INTRODUCTION

The basis for much of smooth chaotic dynamics is the Anosov condition, also referred to as complete, uniform, exponential hyperbolicity. Under this condition, there are unique stable and unstable bundles that are invariant under the dynamics. The bundles are uniquely integrable, and their integral foliations have smooth leaves.

A major technical obstacle to understanding such a dynamical system—for instance, proving that it is ergodic, or calculating how its Lyapunov exponents and entropy vary—arises from the fact that:

Although the invariant bundles are continuous and even obey Hölder conditions, in general they fail to be differentiable.

* Dedicated to David Ruelle on his 65th birthday.

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Correspondingly, the stable and unstable integral foliations fail to be differentiable, despite the fact that their leaves are smooth.[4] If we denote the stable and unstable bundles by E^s and E^u, then:

E^s is differentiable in the E^s direction (because the stable leaves are smooth), but it is generally nondifferentiable in the transverse E^u direction.

Similarly, E^u is differentiable in the E^u direction but not in the E^s direction. Symbolically, we could express this as existence and continuity of the partial derivatives:

$$\frac{\partial E^s}{\partial E^s} \quad \text{and} \quad \frac{\partial E^u}{\partial E^u}. \tag{1}$$

In this paper we generalize to the case that the dynamics are only partially hyperbolic, which means that there is a third invariant bundle, the center bundle, E^c. The dynamics in the center directions are neutral in comparison with the stable and unstable behavior. Under fairly general hypothesis, we show that E^s and E^u are differentiable in the E^c direction. Symbolically, this is expressed as existence and continuity of:

$$\frac{\partial E^s}{\partial E^c} \quad \text{and} \quad \frac{\partial E^u}{\partial E^c}. \tag{2}$$

In the partially hyperbolic case, the stable and unstable leaves are smooth, so (2) can be improved to existence and continuity of

$$\frac{\partial E^s}{\partial E^{cs}} \quad \text{and} \quad \frac{\partial E^u}{\partial E^{cu}}, \tag{3}$$

where $E^{cs} = E^c \oplus E^s$, and $E^{cu} = E^c \oplus E^u$.

One feature to note in what we do below is that we do not assume that the center bundle is integrable. Although integrability of E^c is known to hold in many cases, it is not true when the center's neutrality is insufficiently dominated by the hyperbolicity of the stable and unstable bundles. See ref. 16. Furthermore, even in well-dominated cases, integrability seems to be a subtle issue. While our results do not assume integrability of the center, they do assume "hyperbolic domination"—the hyperbolic part of f dominates the center part.

[4] It was D. V. Anosov who first uncovered and then dealt with this apparently pathological aspect of the stable and unstable foliations.[2] Furthermore, foliations that have smooth leaves but fail to be differentiable in the transverse direction can, in general, have behavior that makes them useless in terms of ergodic theory. See J. Milnor's article,[10] in which he produces an invariant foliation whose leafwise zero sets have full Lebesgue measure.

Partial hyperbolicity is defined in the next section. For the reader unfamiliar with the definition, there are two key examples to keep in mind. The first example is the time-1 map of an Anosov flow—for instance, the geodesic flow on the unit tangent bundle of a negatively curved manifold. For this diffeomorphism, there are three distinguished subspaces of the tangent space at every point—the contracting (or stable) subspace, the expanding (unstable) subspace, and the space spanned by the generating vector field of the flow. These spaces are invariant under the time-one map of the flow. Hyperbolic domination is valid because f is an isometry in the center direction.

The other example to keep in mind is a (potentially) infinite dimensional one. Let $\mathscr{F}$ be a smooth family of Anosov diffeomorphisms of a compact manifold M, such as a smooth one-parameter family of Anosov diffeomorphisms. (More generally, $\mathscr{F}$ could be a smooth family of partially hyperbolic diffeomorphisms.) Associated to any such family is the evaluation map,

$$\mathrm{Eval}\colon \mathscr{F} \times M \to \mathscr{F} \times M$$

$$(f, p) \mapsto (f, f(p)).$$

This map is partially hyperbolic (Proposition 7.1), with a center bundle transverse to the M factor.

Our main result, Theorem A, allows us to address the following questions about these specific systems. For the time-1 map of the Anosov flow, the existence and continuity of the partial derivatives in (2) says that the leaves of the stable (horocyclic) foliation vary differentiably along leaves of the center foliation. Similarly, the unstable foliation is differentiable along leaves of the center foliation. While this fact is well-known and easily seen to hold—because the leaves of the center foliation are orbits of the flow—it is no longer obvious when the time-1 map is slightly perturbed. Any perturbation of this time-1 map remains partially hyperbolic, even though it is no longer necessarily the time-1 map of a flow. Theorem A implies that the unstable and stable foliations remain differentiable along leaves of the new center foliation even though the new center foliation is no longer transversely smooth. Differentiability of the unstable bundle along the center was a crucial ingredient in proving stable ergodicity for many partially hyperbolic diffeomorphisms,[5, 11, 12, 16] starting with the time-1 map of a geodesic flow. It was also an ingredient in the construction of nonuniformly hyperbolic diffeomorphisms with pathological foliations.[4, 14, 15]

For the partially hyperbolic map Eval, the M-component of an integral curve tangent to its center bundle is a "dynamically-defined" curve γ in M, and with enough hyperbolic domination, the unstable bundle of f

can be differentiated along γ. We make this notion precise in Theorem B, which is proved in Section 7. This type of result has been used to show that Kolmogorov–Sinai entropy and SRB states also vary differentiably with parameters for Anosov diffeomorphisms and flows.[7, 8, 13] While we have similar applications in mind for Theorem A, we will content ourselves here with some general theorems. We describe the main results of this paper in the following section; the proofs occupy the remaining sections.

In Section 4 we derive explicit series expansions for the partial derivatives in (2).

2. STATEMENTS OF RESULTS

Partial hyperbolicity is a weakening of the Anosov condition that allows for much more dynamical complexity. A diffeomorphism f is **partially hyperbolic** if the tangent bundle to M splits as a Tf-invariant sum:

$$TM = E^u \oplus E^c \oplus E^s,$$

with at least two of the subbundles in the sum nontrivial, and there exist constants $a < b < 1 < c < d$, and a Finsler structure $|\cdot|$ on M such that, for all $p \in M$ and all $v \in T_pM$,

$$v \in E^u(p) \Rightarrow d\,|v| \leqslant |T_p f(v)|$$
$$v \in E^c(p) \Rightarrow b\,|v| \leqslant |T_p f(v)| \leqslant c\,|v|$$
$$v \in E^s(p) \Rightarrow |T_p f(v)| \leqslant a\,|v|.$$

The bundles E^u, E^c, and E^s are **unstable**, **center**, and **stable** bundles for f, and we write $Tf = T^u f \oplus T^c f \oplus T^s f$ correspondingly. It is not necessary to assume these bundles are continuous in the definition; continuity follows from invariance and the growth conditions given above. A similar argument shows that if the constants a, b, c, d are fixed then the splitting is unique and varies continuously under perturbation of f in the C^1 topology; see, e.g., ref. 6. Equivalent definitions of partial hyperbolicity are given by Brin and Pesin in ref. 3 and by Hirsch, Pugh, and Shub in ref. 6.

Several classically studied dynamical systems are partially hyperbolic, including time-t maps of Anosov flows, frame flows for negatively curved manifolds, and certain algebraic systems.

Recall that the **norm**, **conorm**, and **bolicity** of a linear transformation $T: X \to Y$ on Banach spaces are defined as

$$\|T\| = \sup_{|x|=1} |Tx|, \qquad \mathbf{m}(T) = \inf_{|x|=1} |Tx|, \qquad \mathrm{bol}(T) = \frac{\|T\|}{\mathbf{m}(T)}.$$

Here, $|x|$ and $|Tx|$ are the norms of x and Tx in X and Y. The bolicity is also called the condition number. Partial hyperbolicity can be restated as

$$\sup_p \|T^s_p f\| < \min\{1, \inf_p \mathbf{m}(T^c_p f)\} \quad \text{and} \quad \max\{1, \sup_p \|T^c_p f\|\} < \inf_p \mathbf{m}(T^u_p f),$$

where the infima and suprema are taken over all $p \in M$.

Suppose that $f: M \to M$ is a C^2 partially hyperbolic diffeomorphism with splitting $E^u \oplus E^c \oplus E^s$. In general its summands are continuous but not C^1. Here we show that under hyperbolic dominance conditions, E^u and E^s are continuously differentiable in the E^c direction, i.e., $\partial E^u_p/\partial E^c$ and $\partial E^s_p/\partial E^c$ exist and are continuous functions of $p \in M$. The **hyperbolic dominance conditions** are

$$\sup_p \frac{\mathrm{bol}(T^c_p f)}{\mathbf{m}(T^u_p f)} < 1 \tag{4}$$

and

$$\sup_p \mathrm{bol}(T^c_p f)\, \|T^s_p f\| < 1. \tag{5}$$

These conditions are related to the center bunching hypotheses of refs. 11 and 12 for they amount to saying that the center bolicity is so near 1 that it is overwhelmed by the hyperbolicity of $T^u f \oplus T^s f$.

Theorem A. (4) implies that E^u is continuously differentiable with respect to E^c, and (5) implies that E^s is continuously differentiable with respect to E^c.

Theorem A is a corollary of a more general result about dominated splittings—see Theorem 5.1 in Section 5.

Next we discuss differentiating E^u along special curves in M. Let $\mathcal{PH}$ denote the open subset of partially hyperbolic diffeomorphisms in $\mathrm{Diff}^2(M)$. As we stated in the introduction, and as will be proved in Section 7, the evaluation map

$$\begin{aligned} \mathrm{Eval}: \mathcal{PH} \times M &\to \mathcal{PH} \times M \\ (f, p) &\mapsto (f, f(p)) \end{aligned}$$

is partially hyperbolic. Its center bundle is infinite dimensional, and not necessarily integrable, but there are curves (f_t, p_t) tangent to it. We say that the M-component, p_t, is a **dynamically defined** curve in M.

Example. Let $\mathscr{F}$ be a smooth curve of Anosov diffeomorphisms of M, $\mathscr{F} = \{f_t\}_{-\epsilon<t<\epsilon}$. We check directly that

$$\begin{aligned} \text{Eval}\colon \mathscr{F}\times M &\to \mathscr{F}\times M \\ (f_t, p) &\mapsto (f_t, f_t p) \end{aligned}$$

is partially hyperbolic and its dynamically defined curves p_t are nothing more than the curves $h_t p$ where h_t is the canonical conjugacy from f_0 to f_t. This justifies the name "dynamically defined." By rescaling f_t as $f_{\lambda t}$ where $\lambda > 0$ is small and fixed, we may assume that $\epsilon = 1$ and $\|f_t - f_0\|_{C^1}$ is small, $-1 < t < 1$. Then Eval is a small perturbation of the product $F_0 = \mathrm{id} \times f_0$. Since f_0 is Anosov, F_0 is partially hyperbolic with splitting

$$T(\mathscr{F}\times M) = (0\times E^u) \oplus (T\mathscr{F}\times 0) \oplus (0\times E^s),$$

where the Anosov splitting of f_0 is $TM = E^u \oplus E^s$. Its center foliation $\mathscr{W}_0$ has leaves that are the parallel curves $\mathscr{F}\times p$, $p\in M$, and there is a unique nearby F_λ-invariant foliation $\mathscr{W}_\lambda$. In fact, by Theorem 7.1 of ref. 6, there is a unique "leaf conjugacy"

$$H\colon \mathscr{F}\times M \to \mathscr{F}\times M$$

that sends $\mathscr{W}_0$-leaves to $\mathscr{W}_\lambda$-leaves, is close to the identity, and is of the form

$$H(f_t, p) = (f_t, h(t, p)).$$

Thus, dynamically defined curves are all of the form $h(t, p)$. Invariance implies that

$$F^n(H(\mathscr{F}\times p)) = H(F_0^n(\mathscr{F}\times p))$$

for all $n\in\mathbb{Z}$. Since H is near the identity map, this implies that $p\mapsto h(t, p)$ is the canonical conjugacy h_t from f_0 to its perturbation f_t.

Now we return to the general case in which f_t is a C^2 curve in $\mathscr{PH}$, $-\epsilon < t < \epsilon$, with splitting

$$TM = E_t^u \oplus E_t^c \oplus E_t^s.$$

Suppose that (f_t, p_t) is tangent to the center bundle of Eval, so that p_t is dynamically defined. In this notation, E^u_{t,p_t} refers to the fiber of the bundle E_t^u at the point p_t.

Theorem B. For t near 0, $t \mapsto E^u_{t,p_t}$ is a C^1 curve in the Grassmann of TM provided that Tf_0 satisfies the hyperbolic dominance condition (4). Similarly, (5) implies that $t \mapsto E^s_{t,p_t}$ is C^1.

Theorem B follows from a more general result, Theorem 7.4, proved in Section 7. The machinery behind the proofs of Theorems A and B is Theorem 3.1, a refinement of the C^1 Section Theorem from ref. 5 that handles partial derivatives of a section.

In Section 8, we address the question of when $t \mapsto E^u_t$ is differentiable at $t = 0$. The issue here is of a slightly different nature than that in Theorems A and B. While $t \mapsto E^u_t$ is always continuously differentiable along dynamically defined paths, the requirement that the constant path $t \mapsto p$ be dynamically defined for all p is a stringent one, satisfied only for very special families.

If, instead of requiring that $t \mapsto E^u_t$ be C^1 in a given family, we just ask that it be differentiable at $t = 0$ *but for all families through* f_0, then the actual dynamics of f_0 becomes irrelevant. It is easy to see that E^u_0 must be C^1 for this property to hold. What is interesting is that nonsmoothness of E^u_0 is the *only* obstruction. Building on Theorem B, and using the same notation, one can show:

Theorem C. Suppose that E^u_0 is a $C^{2-\epsilon}$ subbundle of TM, for all $\epsilon > 0$, and that f_0 satisfies the hyperbolic dominance condition (4). Then for all $p \in M$, $t \mapsto E^u_{t,p}$ is differentiable at $t = 0$. Furthermore, for any dynamically defined curve p_t, if v is tangent to p_t at $t = 0$ then

$$E^u_{t,p} - E^u_{0,p} = \left(\frac{d}{dt}\bigg|_{t=0} E^u_{t,p_t} - D_p E^u_0(v) \right) t + O(t^{1+\eta}),$$

for some $\eta > 0$.

Subsequent to proving Theorem C, we learned of a more general result, due to Dolgopyat. Instead of a partially hyperbolic splitting, he assumes a three-way dominated splitting for the curve f_t of diffeomorphisms,

$$TM = R_t \oplus S_t \oplus T_t.$$

Theorem D (Dolgopyat, ref. 4, Theorem 3). If S_0 is a C^1 bundle then for all $p \in M$, $t \mapsto S_{t,p}$ is differentiable at $t = 0$.

The notation $S_{t,p}$ refers to the fiber of S_t at the point p. Dominated splittings are defined in Section 5. In particular, Theorem D applies when S is E^u, E^c, E^s, E^{cu}, or E^{cs}. We present an exposition of Dolgopyat's proof of Theorem D in Section 8.

3. PARTIAL DERIVATIVES OF AN INVARIANT SECTION

Let

$$\begin{array}{ccc} V & \xrightarrow{F} & V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & M \end{array}$$

be a C^1 fiber preserving map, where V is a smooth, finite dimensional fiber bundle over the compact manifold M, and f is a diffeomorphism. In addition assume that there is a section $\sigma: M \to V$, invariant under F in the sense that

$$F(\sigma(p)) = \sigma(f(p))$$

for all $p \in M$.

In general there is no reason that σ is smooth, or even continuous. For example, if F is the identity map, every section of V is F-invariant. In ref. 6, we showed that if V is a Banach bundle and F is a fiber contraction then σ is unique and continuous, and furthermore, if F contracts the fiber sufficiently more sharply than the base then σ is of class C^r.

Since F preserves fibers, TF preserves the "vertical" subbundle, $\text{Vert} \subset TV$ whose fiber at $v \in V$ is kernel $T_v\pi$. We write $T_v^{\text{Vert}}F$ for the restriction of T_vF to Vert_v,

$$T_v^{\text{Vert}}F\colon \text{Vert}_v \to \text{Vert}_{Fv}\,.$$

We assume that TV carries a Finsler structure and that $k_p = \|T_{\sigma p}^{\text{Vert}}F\|$ has

$$\sup_{p \in M} k_p < 1,$$

which means that F is a fiber contraction in the neighborhood of σM.

Theorem 3.1. Suppose that $E \subset TM$ is a continuous Tf-invariant subbundle such that

$$\sup_p \frac{k_p}{\mathbf{m}(T_p^E f)} < 1$$

where $T^E f$ is the restriction of Tf to E. Then σ is continuously differentiable in the E-direction in the sense that there is a continuous map $H: E \to TV$ such that

(a) For each $p \in M$, $H: E_p \to T_{\sigma p} V$ is linear.

(b) $T\pi \circ H = \mathrm{Id}: E \to E$.

(c) If γ is a C^1 arc in M that is everywhere tangent to E then

$$(\sigma \circ \gamma)'(t) = H(\gamma'(t)).$$

In particular, if E is integrable then the restriction of σ to each E-leaf is C^1.

We refer to H as the partial derivative of σ in the E-direction

$$H = \frac{\partial \sigma}{\partial E}.$$

Remark. If, in addition, there exist C^r submanifolds everywhere tangent to E, $1 \leqslant r < \infty$, then C^r smoothness of σ along E (i.e., along these manifolds) is implied by

$$\sup_p \frac{k_p}{\mathbf{m}(T_p^E f)^r} < 1.$$

Remark. When E is integrable, the proof of Theorem 3.1 is a fairly simple application of the Invariant Section Theorem of ref. 6. It is the non-integrable case that requires some new ideas.

Remark. There is a uniformity about $\partial\sigma/\partial E$. (In the integrable case, this uniformity is automatic.) Fix $p \in M$ and extend each $w \in E_p$ with $|w| \leqslant 1$ to a continuous vector field X_w everywhere subordinate to E, and do so in a way that depends continuously on w. Let γ_w be an integral curve of X_w through p. Since E is only continuous, the integral curve γ_w need not be uniquely determined by X_w. Nevertheless, for all p in any fixed C^1 chart, as $t \to 0$ we have

$$\frac{\sigma \circ \gamma_w(t) - \sigma p}{t} \to H(w)$$

uniformly.

Remark. Since M is finite dimensional, Peano's Existence Theorem implies that there exist C^1 arcs everywhere tangent to a continuous plane field, and thus the hypothesis of assertion (c) in Theorem 3.1 is satisfied. In the infinite dimensional case, however, Peano's Theorem fails and (c) could become vacuous.

Proof of Theorem 3.1. We proceed by the graph transform techniques in ref. 6. Choose a continuous subbundle $\mathrm{Hor} \subset TV$, complementary to Vert,

$$\mathrm{Hor} \oplus \mathrm{Vert} = TV.$$

For example, we could introduce a Riemann structure on TV and take Hor_v as the orthogonal complement to Vert_v. Note that $T\pi$ sends each subspace Hor_v isomorphically onto T_pM, $p = \pi v$. With respect to the horizontal/vertical splitting we write

$$T_vF = \begin{bmatrix} A_v & 0 \\ C_v & K_v \end{bmatrix} = \begin{bmatrix} A_v\colon \mathrm{Hor}_v \to \mathrm{Hor}_{Fv} & 0 \\ C_v\colon \mathrm{Hor}_v \to \mathrm{Vert}_{Fv} & K_v\colon \mathrm{Vert}_v \to \mathrm{Vert}_{Fv} \end{bmatrix}.$$

Let $\bar{E} \subset \mathrm{Hor}$ be the lift of E. That is, $T\pi$ sends the plane $\bar{E}_v$ isomorphically to E_p, $p = \pi v$. Since E is Tf-invariant and F covers f, $\bar{E}$ is A-invariant in the sense that

$$\begin{array}{ccc} \bar{E}_v & \xrightarrow{A_v} & \bar{E}_{Fv} \\ {\scriptstyle T\pi}\downarrow & & \downarrow{\scriptstyle T\pi} \\ E_p & \xrightarrow{Tf} & E_{fp} \end{array}$$

commutes.

Let L be the bundle over M whose fiber at p is

$$L_p = L(\bar{E}_{\sigma p}, \mathrm{Vert}_{\sigma p}).$$

An element in L_p is a linear transformation $P\colon \bar{E}_{\sigma p} \to \mathrm{Vert}_{\sigma p}$. Let LF be the **graph transform** on L that sends $P \in L_p$ to

$$P' = (C_{\sigma p} + K_{\sigma p}P)(A_{\sigma p}|_{\bar{E}_{\sigma p}})^{-1} \in L_{fp}.$$

Then TF sends the graph of P to the graph of P' and LF is an affine fiber contraction

$$\begin{array}{ccc} L & \xrightarrow{LF} & L \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

By Theorem 3.1 in ref. 6, L has a unique LF-invariant section $\Lambda: M \to L$, and Λ is continuous. Define $H_p: E_p \to T_{\sigma p}V$ by commutativity of

$$\begin{array}{ccc} \bar{E}_{\sigma p} & \xrightarrow{\mathrm{Id}_p \oplus \Lambda_p} & \bar{E}_{\sigma p} \oplus \mathrm{Vert}_{\sigma p} \\ T\pi \downarrow & & \downarrow \text{Inclusion} \\ E_p & \xrightarrow{H_p} & T_{\sigma p}V \end{array}$$

where Id_p is the identity map $\bar{E}_{\sigma p} \to \bar{E}_{\sigma p}$. Then $H: E \to TV$ is the unique bundle map such that HE is a TF-invariant subbundle of $T_{\sigma M}V$,

$$\begin{array}{ccc} HE & \xrightarrow{TF} & HE \\ T\pi \downarrow & & \downarrow T\pi \\ E & \xrightarrow{Tf} & E \end{array}$$

commutes, and $T\pi \circ H = \mathrm{Id}_E$. We claim that H is the partial derivative of σ in the E-direction.

Let $\gamma: (a, b) \to M$ be a C^1 arc such that γ is everywhere tangent to E. To complete the proof of the theorem, we must show that

$$(\sigma \circ \gamma)'(t) = H(\gamma'(t)).$$

For $n \in \mathbb{Z}$, set $\gamma_n = f^n \circ \gamma$ and

$$\Gamma = \bigsqcup_{n \in \mathbb{Z}} \gamma_n.$$

This means that we consider the disjoint union of the arcs γ_n, so if two of them cross in M, we ignore the crossing in Γ. The one dimensional manifold Γ is noncompact; it has countably many components γ_n. In the same way, we discretize V as

$$V_\Gamma = \bigsqcup_n V|_{\gamma_n}.$$

We equip V_Γ and $T\Gamma$ with the Finslers they inherit from V and M. Then $F_\Gamma = F|_{V_\Gamma}$ is a fiber contraction

$$\begin{array}{ccc} V_\Gamma & \xrightarrow{F_\Gamma} & V_\Gamma \\ \pi \downarrow & & \downarrow \pi \\ \Gamma & \xrightarrow{f} & \Gamma \end{array}$$

and F_Γ contracts the fiber more sharply than the base since

$$\sup_p \frac{k_p}{\mathbf{m}(T_p^E f)} < 1$$

and $T\Gamma \subset E$. Furthermore, F_Γ is uniformly C^1 bounded since M is compact. The Invariant Section Theorem (ref. 6, Theorem 3.2) then implies that V_Γ has a unique F_Γ-invariant section σ_Γ, and σ_Γ is of class C^1. Furthermore the tangent bundle of $\sigma_\Gamma(\Gamma)$ is the unique nowhere vertical TF_Γ-invariant line field in TV_Γ.

The restriction of σ to $\Gamma = \bigsqcup_n \gamma_n$ is F_Γ-invariant, so by uniqueness

$$\sigma_\Gamma = \bigsqcup_n \sigma|_{\gamma_n}.$$

We claim that

$$H(T\Gamma) = T(\sigma_\Gamma \Gamma).$$

Again the reason is uniqueness. We know that $T(\sigma_\Gamma \Gamma)$ is the unique TF_Γ-invariant, nowhere vertical line field defined over $\sigma_\Gamma\Gamma$. But commutativity of

$$\begin{array}{ccc} HE & \xrightarrow{TF_\Gamma} & HE \\ H \uparrow & & \uparrow H \\ E & \xrightarrow{Tf} & E \\ \text{Inclusion} \uparrow & & \uparrow \text{Inclusion} \\ T\Gamma & \xrightarrow{Tf} & T\Gamma \end{array}$$

implies that $H(T\Gamma)$ is a second such line field. By uniqueness they are equal.

To complete the proof, we show that the line field equality implies the vector equality

$$\frac{d}{dt}\,\sigma \circ \gamma(t) = H(\gamma'(t)),$$

as the theorem asserts. Differentiating $\gamma(t) = \pi \circ \sigma_\Gamma \circ \gamma(t)$ gives

$$\gamma'(t) = T\pi \circ T\sigma_\Gamma(\gamma'(t)).$$

The vector $T\sigma_\Gamma(\gamma'(t))$ lies in the span of $H(\gamma'(t))$, so there is a real number $c(t)$ such that $T\sigma_\Gamma(\gamma'(t)) = H(c(t)\,\gamma'(t))$. This gives

$$\gamma'(t) = T\pi \circ H(c(t)\,\gamma'(t)).$$

Since $T\pi \circ H = \mathrm{Id}_E$ we have

$$\gamma'(t) = c(t)\,\gamma'(t)$$

and $c(t) = 1$. Thus

$$\frac{d}{dt}\,\sigma_\Gamma \circ \gamma(t) = T\sigma_\Gamma(\gamma'(t)) = H(c(t)\,\gamma'(t)) = H(\gamma'(t)). \quad \blacksquare$$

Remark. Above, it is assumed that γ is everywhere tangent to E. One might expect that tangency of γ to E at $p = \gamma(0)$ suffices to prove that $(\sigma \circ \gamma)'(0) = H(\gamma'(0))$. This is not so. For example E can be the flow direction of an Anosov flow. The bundle E^u can be Hölder, but not C^1. Say its Hölder exponent is $\theta < 1$. One can construct a C^1 curve $\gamma(t)$ which is tangent to E at $p = \gamma(0)$, but which diverges from E at a rate $t^{1+\epsilon}$. The difference between $E^u_{\gamma(t)}$ and E^u_p is then on the order of $t^{\theta+\epsilon\theta}$. If ϵ is small this exponent is < 1, and the map $t \mapsto E^u_{\gamma(t)}$ fails to be differentiable at $t = 0$.

4. A SERIES EXPRESSION FOR $\partial\sigma/\partial E$

As above σ is the unique F-invariant section and $H = \mathrm{Id}_E \oplus \Lambda$ is its partial derivative in the E-direction. Naturally, $\partial\sigma/\partial E$ depends on the choice of horizontal subbundle $\mathrm{Hor} \subset TV$. We use the isomorphism $T\pi\colon \mathrm{Hor}_{\sigma p} \to T_pM$ to identify the linear map $A_{\sigma p}\colon \mathrm{Hor}_{\sigma p} \to \mathrm{Hor}_{\sigma(fp)}$ with its $T\pi$-conjugate $T_p f$. Then, using the canonical isomorphism $\mathrm{Vert}_{\sigma p} \approx V_p$, we can express $TF = \begin{bmatrix} A & 0 \\ C & K \end{bmatrix}$ as

$$T_{\sigma p}F = \begin{bmatrix} T_p f\colon T_pM \to T_{fp}M & 0 \\ C_p\colon T_pM \to V_{fp} & K_p\colon V_p \to V_{fp} \end{bmatrix}.$$

Thus, the bundle map $LF\colon L \to L$ becomes

$$P \mapsto (C_p + K_p P) \circ (T^E_{fp} f^{-1}).$$

Denote by Λ_0 the zero section of L, and call its N^{th} iterate in L,

$$\Lambda_N = (LF)^N (\Lambda_0).$$

We know that $\Lambda_N \to \Lambda$ uniformly as $N \to \infty$. Also, we claim that

$$\Lambda_N(p) = \sum_{n=0}^{N-1} K^n_p \circ C_{f^{-n-1}(p)} \circ (T^E_p f^{-n-1})$$

where $K^0 = \text{Id}$ and for $n \geqslant 1$,

$$K^n_p = K_{f^{-1}(p)} \circ \cdots \circ K_{f^{-n}(p)}\colon V_{f^{-n}(p)} \to V_p.$$

If $N = 1$ we have

$$\Lambda_1(p) = (C_{f^{-1}(p)} + K_{f^{-1}(p)} P_0)(T^E_p f^{-1}) = C_{f^{-1}(p)} T^E_p f^{-1}$$

because $\Lambda_0 = 0$ implies that $P_0 = 0$. Thus, the assertion holds with $N = 1$; the proof is completed by induction.

Since the partial sums of the infinite series $\sum_{n=0}^{\infty} K^n_p C_{f^{-n-1}(p)} T^E_p f^{-n-1}$ converge uniformly to Λ, we are justified in writing

$$\frac{\partial \sigma}{\partial E} = H(p) = \text{Id}_E \oplus \sum_{n=0}^{\infty} K^n_p C_{f^{-n-1}(p)} T^E_p f^{-n-1}.$$

5. PROOF OF THEOREM A

In this section we consider a generalization of partial hyperbolicity to so-called dominated splittings, and we prove Theorem A in the more general context. We first review the definitions. If $T = T_1 \oplus T_2$ with respect to $X = X_1 \oplus X_2$ then hyperbolicity means that

$$\|T_2\| < 1 < \mathbf{m}(T_1).$$

A weaker condition is that

$$\text{dom}(T) = \frac{\|T_2\|}{\mathbf{m}(T_1)} < 1,$$

in which case we say that T_1 **dominates** T_2, because the former stretches more than the latter. We write $T_1 > T_2$. The constant $\mathrm{dom}(T)$ quantifies the degree of domination; the smaller the domination constant, the stronger the domination. We say that a splitting $TM = R \oplus S$ is **dominated** for a diffeomorphism $f: M \to M$ if

(a) R and S are continuous Tf-invariant subbundles of TM.
(b) There is a Finsler on TM such that for all $p \in M$, $T_p^R f > T_p^S f$,

where the notation $T^X f$ indicates the restriction of Tf to a subbundle X. (Compactness implies that $\sup_p \|T_p^S f\| / \mathbf{m}(T_p^R f) < 1$.)

Remark. In Chapter 5 of ref. 6 we introduced the concept of a dominated splitting, calling it pseudo-hyperbolicity. Although the term did not catch on, it is grammatically superior to "dominated." There is nothing that dominates the splitting or the tangent map in a dominated splitting. Rather, one summand of the tangent map dominates the other. Perhaps one should speak of a de Sade splitting; there is a dominator and a dominated.

If $TM = E^u \oplus E^c \oplus E^s$ is a partially hyperbolic splitting for $f: M \to M$ then there are two natural dominated splittings

$$R \oplus S = E^u \oplus (E^c \oplus E^s) \qquad \text{and} \qquad R \oplus S = (E^u \oplus E^c) \oplus E^s,$$

but there are others when the bundles E^u, E^c, E^s split into dominated subbundles. Conversely, if $TM = E^u \oplus E^c \oplus E^s$ is Tf-invariant, the two preceding splittings are dominated, and

$$\sup_p \|T_p^s f\| < 1 < \inf_p \mathbf{m}(T_p^u f)$$

then f is partially hyperbolic.

Theorem 5.1. Suppose that $TM = R \oplus S$ is a dominated splitting for the C^2 diffeomorphism $f: M \to M$, and $E \subset TM$ is a Tf-invariant subbundle. If

$$\sup_p \frac{\mathrm{dom}(T_p f)}{\mathbf{m}(T_p^E f)} < 1, \tag{6}$$

then R is continuously differentiable with respect to E. (That is, $\partial R / \partial E$ exists and is continuous.)

Theorem A is an immediate corollary of Theorem 5.1, where we set $R = E^u$, $S = E^c \oplus E^s$, and $E = E^c$.

Proof of Theorem 5.1. Let d be the fiber dimension of R in the dominated splitting

$$TM = R \oplus S.$$

Tf acts naturally on the Grassmann $G = G(d, TM)$ of all d-planes in TM,

$$Gf : G \to G.$$

If Π is a d-plane in T_pM then $Gf(\Pi) = T_p f(\Pi)$. Since f is C^2, Gf is a C^1 fiber preserving map,

$$\begin{array}{ccc} G & \xrightarrow{Gf} & G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & M \end{array}$$

where π sends $\Pi \subset T_pM$ to p. Besides, $p \mapsto R_p$ is a Gf-invariant section of G.

A compact neighborhood N_p of R_p in G_p consists of d-planes Π such that $\Pi =$ graph P where P: $R_p \to S_p$ is a linear transformation with $\|P\| \leqslant 1$. Give G a Finsler which is the operator norm on each N_p and any other Finsler on the rest of G. Then Gf is a fiber preserving map whose fiber contraction rate at R_p is

$$k_p = \frac{\|T^S_p f\|}{\mathbf{m}(T^R_p f)} < 1.$$

Since $k_p = \operatorname{dom} T_p f$, Gf contracts the fiber more sharply than f contracts along E, Theorem 3.1 applies and $p \mapsto R_p$ is seen to be a continuously differentiable function of p in the E direction. ▮

6. A SERIES FORMULA FOR $\partial E^u/\partial E^c$

From Section 4 we know that there is a series that expresses $\partial E^u/\partial E^c$. We write this formula out after making a convenient choice of the horizontal bundle.

To do so, we coordinatize G near E^u_p as follows. Fix a smooth Riemann structure on M that exhibits the partial hyperbolicity of f, and let exp be its exponential map. Abusing notation, we denote by $\mathbb{R}^u$ and $\mathbb{R}^{cs}$ the planes $\mathbb{R}^u \times 0$ and $0 \times \mathbb{R}^{cs}$ in $\mathbb{R}^m$. For each $p \in M$, define a linear map I_p: $\mathbb{R}^m \to T_pM$ that carries $\mathbb{R}^u$ and $\mathbb{R}^{cs}$ isometrically to E^u_p and E^{cs}_p. The restriction of $\exp_p \circ I_p$ to a small neighborhood $U = U_p$ of 0 in $\mathbb{R}^m$ is a diffeomorphism of U to a neighborhood $Q = Q_p$ of p in M,

$$\varphi_p : U \to Q$$

and

(a) $\varphi_p(0) = p$

(b) $T_0\varphi_p$ carries $\mathbb{R}^u$ and $\mathbb{R}^{cs}$ isometrically to E^u_p and E^{cs}_p.

(c) If we denote by E^u_{pq} and E^{cs}_{pq} the planes $T_x\varphi_p(\mathbb{R}^u)$ and $T_x\varphi_p(\mathbb{R}^{cs})$, where $q = \varphi_p(x)$, then $q \mapsto E^u_{pq} \oplus E^{cs}_{pq}$ is a smooth splitting of TQ that reduces to $E^u_p \oplus E^{cs}_p$ when $p = q$.

We now coordinatize G near E^u_p. Let $\mathscr{M}$ be the space of $(u \times cs)$-matrices, thought of as linear transformations $X: \mathbb{R}^u \to \mathbb{R}^{cs}$. Given $(x, X) \in U \times \mathscr{M}$, let $q = \varphi_p(x)$ and consider the linear transformation $S: E^u_{pq} \to E^{cs}_{pq}$, defined by commutativity of

$$\begin{tikzcd} \mathbb{R}^u \arrow[r, "X"] \arrow[d, "T_x\varphi_p"'] & \mathbb{R}^{cs} \arrow[d, "T_x\varphi_p"] \\ E^u_{pq} \arrow[r, "S"] & E^{cs}_{pq} \end{tikzcd}.$$

The graph of S is a plane $\Pi \in G$ near E^u_p, and thus

$$\Phi_p : (x, X) \mapsto \Pi$$

is a local trivialization of G at E^u_p.

Because $U \times \mathscr{M}$ is a product, $T(U \times \mathscr{M})$ carries a natural horizontal structure, the horizontal space at (x, X) being

$$\mathbb{R}^m \times 0 \subset \mathbb{R}^m \times \mathscr{M} = T_{(x, X)}(U \times \mathscr{M}).$$

We define the horizontal space at $\Pi = \Phi_p(0, X) \in G_p$ to be

$$\mathrm{Hor}_\Pi = T_{(0, X)}\Phi_p(\mathbb{R}^m \times 0).$$

Writing $T(Gf): TG \to TG$ with respect to the horizontal/vertical splitting of TG gives

$$T_\Pi(Gf) = \begin{bmatrix} A_\Pi : \mathrm{Hor}_\Pi \to \mathrm{Hor}_{Gf(\Pi)} & 0 \\ C_\Pi : \mathrm{Hor}_\Pi \to \mathrm{Vert}_{Gf(\Pi)} & K_\Pi : \mathrm{Vert}_\Pi \to \mathrm{Vert}_{Gf(\Pi)} \end{bmatrix}.$$

Take $\Pi = E^u_p$ and identify

$$T_{E^u_p} G_p = \mathrm{Vert}_{E^u_p} \approx L(E^u_p, E^{cs}_p).$$

Fix $v \in E^c_p$. Then $C_{f^{-n-1}p} \circ T^c f^{-n-1}(v)$ is a linear transformation $Y_n(v)$: $E^u_{f^{-n}p} \to E^{cs}_{f^{-n}p}$, and $Y_n(v)$ is susceptible to the nth power of the linear graph transform, which converts it to a linear transformation $E^u_p \to E^{cs}_p$ defined by

$$T^{cs}_{f^{-n}p} f^n \circ (Y_n(v)) \circ T^u_p f^{-n}.$$

This is the same as the repeated action of K. (That is, the graph transform of Tf^n is the same as the n^{th} power of the graph transform of Tf.) Thus, by the formula in Section 4,

$$\frac{\partial E^u}{\partial E^c}(v) = \sum_{n=0}^{\infty} T^{cs}_{f^{-n}p} f^n \circ (C_{f^{-n-1}p} \circ T^c f^{-n-1}(v)) \circ T^u_p f^{-n}.$$

We also can express this in charts as follows. Writing f in the φ-charts gives

$$f_p = \varphi^{-1}_{fp} \circ f \circ \varphi_p$$

and

$$(Df_p)_x = \begin{bmatrix} D^{u,u}_x f_p & D^{cs,u}_x f_p \\ D^{u,cs}_x f_p & D^{cs,cs}_x f_p \end{bmatrix}$$

where the $D^{u,u}_x f_p$ block consists of the partial derivatives of the u-components of f_p with respect to the u-variables, evaluated at the point x, etc. At $x = 0$, the off-diagonal blocks are zero, while the diagonal blocks are $T\varphi$-conjugate to $T^u_p f$ and $T^{cs}_p f$. Thus, the coordinate expression of Gf becomes

$$(Gf)_p\colon (x, X) \mapsto (f_p x, (D^{u,cs}_x f_p + (D^{cs,cs}_x f_p)\, X)(D^{u,u}_x f_p + (D^{cs,u}_x f_p) X)^{-1}.$$

Differentiating this with respect to x and X at the origin $(0, 0) \in \mathbb{R}^m \times \mathscr{M}$ yields

$$(D((Gf)_p))_{(0,0)} = \begin{bmatrix} A_p\colon \mathbb{R}^m \to \mathbb{R}^m & 0 \\ C_p\colon \mathbb{R}^m \to \mathscr{M} & K_p\colon \mathscr{M} \to \mathscr{M} \end{bmatrix},$$

where A_p is $T\varphi$-conjugate to $T_p f$,

$$A_p = (T_0 \varphi_{fp})^{-1} \circ T_p f \circ T_0 \varphi_p,$$

C_p represents the second derivatives of f in the φ-charts,

$$\begin{aligned}
C_p &= \frac{\partial}{\partial x}\bigg|_{x=0} (D_x^{u,cs} f_p)(D_x^{u,u} f_p)^{-1} \\
&= (D_x(D_x^{u,cs} f_p))(D_x^{u,u} f_p)^{-1} \\
&\quad -(D_x^{u,cs} f_p)(D_x^{u,u} f_p)^{-1}(D_x(D_x^{u,u} f_p))(D_x^{u,u} f_p)^{-1},
\end{aligned}$$

and, because the off-diagonal blocks vanish at the origin, K_p is $T\Phi$-conjugate to the graph transform of Tf,

$$P \mapsto T_p^{cs} f \circ P \circ (T_p^u f)^{-1}.$$

It is worth noting that the norm of C is uniformly bounded on a neighborhood of E^u in G because f is C^2 and M is compact. Also, this is clear from the formula expressing C in the φ-charts.

7. DEPENDENCE OF E^u, E^s ON f: PROOF OF THEOREM B

As has been highlighted in the Katok–Milnor examples,[10] the conjugacy between an Anosov diffeomorphism and its perturbations is a smooth function of the perturbation, even though the conjugacies themselves are only continuous. As was explained in Section 2, a 1-parameter family of Anosov diffeomorphisms f_t gives rise to a foliation $\mathscr{W}$ whose leaves are C^1 graphs of the form $(f_t, h_t p)$, h_t being the conjugacy from f_0 to f_t. Theorem 3.1 applies perfectly well to this situation, and we conclude that $E^u_{h_t p}$, $E^s_{h_t p}$ are C^1 functions of t. Theorem B replaces the Anosov condition by partial hyperbolicity.

As above, we denote by $\mathscr{PH}$ the open subset of $\mathrm{Diff}^2(M)$ consisting of partially hyperbolic diffeomorphisms.

Theorem 7.1. Eval: $\mathscr{PH} \times M \to \mathscr{PH} \times M$ is partially hyperbolic.

Proof. We denote the splitting for $f \in \mathscr{PH}$ at p as

$$T_p M = E^u_{f,p} \oplus E^c_{f,p} \oplus E^s_{f,p}.$$

By the usual linear graph transform techniques, the splitting depends continuously on f.

The space $\mathrm{Diff}^2(M)$ is a Banach manifold, and its tangent space at f has a natural description; see ref. 1 for details. Let X_f be the Banach space of C^2 sections of the pullback bundle f^*TM, that is, the bundle whose fiber over $p \in M$ is $T_{fp}M$. We write $f+g$ to indicate the diffeomorphism $\exp_f \circ g$. That is, if g is a small vector field in X_f, then

$$(f+g)(p) = \exp_f(g(p))$$

is in $\mathrm{Diff}^2(M)$ and is close to f. So a small disk in X_f is a chart for a small neighborhood of f, and X_f is thereby identified with the tangent space $T_f\mathrm{Diff}^2(M)$.

The tangent to Eval at (f, p) acts on a vector $\begin{bmatrix} g \\ v \end{bmatrix} \in T_{f,p}(\mathscr{PH} \times M)$ as

$$T_{f,p}\mathrm{Eval}\begin{bmatrix} g \\ v \end{bmatrix} = \begin{bmatrix} g \\ g(p) + T_p f(v) \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_{\mathscr{PH}} & 0 \\ \mathrm{ev}_p & T_p f \end{bmatrix}\begin{bmatrix} g \\ v \end{bmatrix},$$

where ev_p evaluates the section of f^*TM at p. In particular, this implies that the subbundles $\mathbf{E}^u = 0 \times E^u$, $\mathbf{E}^s = 0 \times E^s$ are T Eval-invariant. (The bundle $0 \times E^c$ is also T Eval-invariant, but it is too small to be the $\mathbf{E}^c$ we want.)

Since the splitting for $f \in \mathscr{PH}$ depends continuously on f, the hyperbolic parts of the T Eval splitting are continuous. Note that the subbundle $T(\mathscr{PH}) \oplus 0$ is not T Eval-invariant, nor is the subbundle $T(\mathscr{PH}) \oplus E^c$ whose fiber at (f, p) is $X_f \oplus E^c_{f,p}$. For if $v \in E^c_{f,p}$ then the T Eval-image of $\begin{bmatrix} g \\ v \end{bmatrix}$ is $\begin{bmatrix} g \\ g(p) + T^c f_p(v) \end{bmatrix}$, and this vector need not lie in $T(\mathscr{PH}) \oplus E^c$. Nevertheless, partial hyperbolicity implies that the T Eval graph transform defines a fiber contraction of the bundle whose fiber at (f, p) is

$$L(X_f \oplus E^{cu}_{f,p}, E^s_{f,p}),$$

where $E^{cu} = E^u \oplus E^c$. The resulting invariant section is the unique T Eval-invariant subbundle $\mathbf{E}^{cu} \subset T(\mathscr{PH} \times M)$ whose fiber at (f, p) projects isomorphically onto $X_f \oplus E^{cu}_{f,p}$.

Similarly, we find the unique $T\,\mathrm{Eval}^{-1}$-invariant subbundle $\mathbf{E}^{cs} \subset T(\mathscr{PH} \times M)$ whose fiber at (f, p) projects isomorphically onto $X_f \oplus E^{cs}_{f,p}$. Intersecting these bundles, we obtain the T Eval-invariant subbundle $\mathbf{E}^c$. ▌

At the end of this section, we give a series expression for $\mathbf{E}^{cu}$.

It follows from Theorems 3.1 and 7.1 that hyperbolic dominance for a partially hyperbolic f_0 implies that E^u and E^s are continuously differentiable along $\mathbf{E}^c$, the center bundle of Eval, but the following result gives is a useful and more general circumstance in which this occurs.

Corollary 7.2. If $f_0 \in \mathscr{PH}$ has a dominated splitting $R_0 \oplus S_0$ (as well as its partially hyperbolic splitting $E_0^u \oplus E_0^c \oplus E_0^s$) and if it satisfies the hyperbolic domination condition

$$\sup_p \frac{\operatorname{dom}(T_p f_0)}{\mathbf{m}(T_p^c f_0)} < 1, \tag{7}$$

then all f in some neighborhood $\mathscr{F}$ of f_0 have dominated splittings $R_f \oplus S_f$, and R_f is continuously differentiable with respect to $\mathbf{E}^c$, the center bundle of Eval.

Proof of Corollary 7.2. Recall that $\operatorname{dom}(T_p f)$ is $\|T_p^S f\|/\mathbf{m}(T_p^R f)$. Clearly for a small $\mathscr{F}$, we have $TM = E_f^u \oplus E_f^c \oplus E_f^s = R_f \oplus S_f$. We first consider the bundle over $\mathscr{F} \times M$ whose fiber over (f, p) is the space of linear maps $L(R_p(f), S_p(f))$. Since Eval preserves the factors $\{f\} \times M$, its tangent map T Eval induces a graph transform map on this bundle, covering Eval, which is a fiber contraction, with

$$k_{f,p} = \operatorname{dom}(T_p f).$$

The unique invariant section of this graph transform is $\mathbf{R} = 0 \oplus R \subset T\mathscr{F} \times TM$.

Now suppose that γ is any curve tangent to $\mathbf{E}^c$. Such a curve exists, since we can always restrict attention to a finite parameter family of diffeomorphisms. As in the proof of Theorem 3.1, we obtain differentiability of $\mathbf{R}$ (and hence, of R) along γ when $k_{f,p}$ is less than the contraction along γ at (f, p). The latter is bounded below by the conorm of $T_{f,p}$Eval, restricted to $\mathbf{E}^c$, which is the minimum of 1 and $\mathbf{m}(T^c f)$. The hyperbolic dominance condition (7) plus the fact that $k_{f,p} < 1$ imply that for small $\mathscr{F}$, this minimum is larger than $k_{f,p}$, so Theorem 3.1 applies, and we conclude that $\mathbf{R}$ is continuously differentiable along $\mathbf{E}^c$.

Note that Theorem 3.1 needs to be re-proved in this more general infinite dimensional context, but because its original proof relied on uniform estimates (this was the only necessity for the compactness assumption on M), it is not hard to do. ∎

Theorem B involves differentiating along dynamically defined curves, so we first prove they exist.

Proposition 7.3. Dynamically defined curves exist. In particular, if $\mathscr{F}$ is a C^1 curve in $\mathscr{PH}$ then there is a dynamically defined curve p_t through $p \in M$, and its initial E^c component can be prescribed.

Proof. Let $f_t \in \mathscr{F}$ have splitting $E_t^u \oplus E_t^c \oplus E_t^s$, and let $v \in E_{0,p}^c$ be given. Choose a continuous vector field V on M that is subordinate to E_0^c and has $V(p) = v$. (If $v = 0$ we can choose $V = 0$.)

We identify the manifold $\mathscr{F} \times M$ with $(-1, 1) \times M$ in the obvious way. Thus, $T(\mathscr{F} \times M) = \mathbb{R} \times TM$. The center bundle $\mathbf{E}^c$ of Eval at (t, q) is given as the graph of a linear map

$$P_{t,q} \colon \mathbb{R} \times E_{0,q}^c \to E_{0,q}^{us},$$

where $E^{us} = E^u \oplus E^s$. Define a vector field Ω on $(-1, 1) \times M$ by

$$\Omega(t, q) = \frac{\partial}{\partial t} + V(q) + P_{t,q}\left(\frac{\partial}{\partial t} + V(q)\right).$$

Peano's Theorem implies that Ω has an integral curve through $(0, p)$, (t, p_t). The component of its initial tangent in TM is

$$V(p) + P_{0,p}(V(p)) \in E_{0,p}^c \oplus E_{0,p}^{us},$$

so its initial E^c component is $V(p) = v$. Since $\Omega(t, q) \in \mathbf{E}_{t,q}^c$, for all $(t, q) \in (-1, 1) \times M$, p_t is a dynamically defined curve in M. ▮

As mentioned in the introduction, Theorem B is a corollary of the following more general result.

Theorem 7.4. Suppose that f_t is a C^1 curve in $\mathscr{PH}$ and f_t has a dominated splitting $R_t \oplus S_t$. If f_0 satisfies the hyperbolic dominance condition (7) and p_t is a dynamically defined curve in M then for small t, $t \mapsto R_{t,p_t}$ is C^1.

Proof. This is an immediate consequence of Corollary 7.2 and Proposition 7.3. ▮

Remark. E^c is allowed to be the trivial bundle in Theorem 7.4, in which case f_0 is Anosov. If f_0 is Anosov, then, as explained in Section 2, $\mathbf{E}^c$ is uniquely integrable, and $p \mapsto p_t$ is the homeomorphism conjugating f_0 to f_t.

Remark. Similarly, if E^c is integrable and tangent to a plaque-expansive foliation $\mathscr{W}^c$, then $\mathbf{E}^c$ is also tangent to a foliation $\boldsymbol{\mathscr{W}}^c$. See ref. 6, Chapter 7 for a discussion of plaque expansivity. Any center foliation that is C^1 is automatically plaque expansive (ref. 6, Theorem 7.2).

Remark. If f_0 is r-normally-hyperbolic:

$$\|T^c f\|^r < \mathbf{m}(T^u f) \qquad \|T^s f\| < \mathbf{m}(T^c f)^r, \tag{8}$$

then the leaves of $\mathscr{W}^c$ are C^r. In this case, there exist C^r dynamically defined curves p_t. If, in addition, the stronger hyperbolic dominance condition

$$\sup_p \frac{\|T^c_p f_0\|}{\mathbf{m}(T^u_p f_0)\,\mathbf{m}(T^c_p f_0)^r} < 1 \tag{9}$$

holds, then the C^r Section Theorem implies that E^u is C^r along the leaves of $\mathscr{W}^c$ (and so $t \mapsto E^u_{t,p_t}$ is also C^r).

Remark. A simple refinement of the proof shows that $t \mapsto p_t$ and $t \mapsto R_{t,p_t}$ are $C^{1+\alpha}$, where there is a bound on the α-Hölder norm of the t-derivative that is uniform in p, v. The exponent α is determined by several dominance conditions.

7.1. A Series Expansion for E^{cu}

We give a series expression for $\mathbf{E}^{cu}$ as follows. Define the linear map $P^{cu}_{f,p}\colon X \oplus E^{cu}_{f,p} \to E^s_{f,p}$ as the series

$$P^{cu}_{f,p}(g, v) = \sum_{k=0}^{\infty} T^s_{f^{-k}p} f^k(g^s(f^{-k}p)).$$

Note that the series does not depend on v. The domination conditions imply that the series converges. Under T Eval, the graph of $P^{cu}_{f,p}$ is sent to the graph of $P^{cu}_{f,fp}$. Hence, by uniqueness,

$$\mathbf{E}^{cu}_{f,p} = \mathrm{graph}(P^{cu}_{f,p}).$$

In the same way we get a unique T Eval-invariant subbundle $\mathbf{E}^{cs} \subset T(\mathscr{F} \times M)$ whose fiber at (f, p) projects isomorphically onto $X \oplus E^{cs}_{f,p}$, and $\mathbf{E}^{cs}_{f,p} = \mathrm{graph}(P^{cs}_{f,p})$ where

$$P^{cs}_{f,p}(g, v) = \sum_{k=0}^{\infty} T^u_{f^k p} f^{-k}(g^u(f^k p)).$$

The intersection of these two subbundles is the center bundle $\mathbf{E}^c$. Namely, at (f, p), the fiber of $\mathbf{E}^c$ is the graph of the map $P^c_{f,p}\colon X \oplus E^c_{f,p} \to (E^u_{f,p} \oplus E^s_{f,p})$, where

$$P^c_{f,p}(g, v) = \sum_{k=0}^{\infty} T^u_{f^k p} f^{-k}(g^u(f^k p)) + \sum_{k=0}^{\infty} T^s_{f^{-k}p} f^k(g^s(f^{-k}p)).$$

8. WHEN $f \mapsto E^u_f$ ACTUALLY *IS* DIFFERENTIABLE

We described at the beginning of Section 7 how $f \mapsto E^u_f$ is generally not differentiable, even if f is Anosov. In fact, if $p \mapsto E^u_{f_0, p}$ fails to be differentiable in even one direction at p_0, then $f \mapsto E^u_f$ is not differentiable at f_0. For in that case, it is easy to construct a smooth 1-parameter family of diffeomorphisms $\phi_t: M \to M$ such that $t \mapsto E^u_{f_0, \phi_t p_0}$ is not differentiable at $t = 0$; but then, for $g_t = (\phi_t \circ f_0 \circ \phi_t^{-1})$, $E^u_{g_t, p_0} = T\phi_t(E^u_{f_0, \phi_t p_0})$ is not differentiable at $t = 0$.

It turns out that, under the usual hyperbolic dominance hypothesis, nonsmoothness of $p \mapsto E^u_{f_0}$ is the *only* obstruction to differentiability of $f \mapsto E^u_f$ at f_0 in all directions.

The results that follow apply to 1-parameter families of C^2 diffeomorphisms $\{f_t\}_{t \in I}$ such that $t \mapsto f_t$ is a C^1 map from I into $\mathrm{Diff}^2(M)$—a C^1 family of C^2 diffeomorphisms. Since the original proof of Theorem C is somewhat lengthy and the result is subsumed by Theorem D, we omit the proof of Theorem C and present instead a proof of Theorem D, following closely the approach of Dolgopyat in ref. 4.

Assume that for each $t \in I$, f_t is partially hyperbolic with splitting

$$TM = E^u_t \oplus E^c_t \oplus E^s_t.$$

Write

$$E_t = E^c_t \qquad H_t = E^u_t \oplus E^s_t.$$

Theorem 8.1. (Theorem D). If E_0 is a C^1 bundle then the curve of subbundles $t \mapsto E_t$ is differentiable with respect to t at $t = 0$, and the derivative $(dE_t(x)/dt)_{t=0}$ depends continuously on $x \in M$.

Remark. Theorem D remains valid, and the proof is the same, if the partially hyperbolic splitting is replaced by a dominated triple splitting $R_t \oplus S_t \oplus T_t$. Namely, the middle bundle S_t is differentiable with respect to t at $t = 0$, provided that $S_{x,0}$ is C^1. Similarly, there is nothing special about the one-dimensionality of the parameter t.

The following facts about weak continuity will be used. We assume that W is a Banach space, but that W also carries a weak topology. Of course, if W has finite dimension, the weak and strong topologies coincide. We have in mind the case that W is a space of operators on the the infinite dimensional Banach space of continuous sections of a vector bundle and $\Lambda = \mathbb{R}$.

Definition 8.2. A function $h: \Lambda \to W$ is **weakly continuous** at $\mu \in \Lambda$ if $h(\lambda)$ tends weakly to $h(\mu)$ and $\|h(\lambda)\|$ stays bounded as $\lambda \to \mu$.

Proposition 8.3. (Weak Inversion Rule). If a curve of invertible operators $t \mapsto A_t$ is weakly continuous at $t = 0$ and if the operators' conorms are uniformly positive then the curve of inverse operators is also weakly continuous at $t = 0$.

Proof. Let $t \mapsto A_t$ be the curve of operators, and let V be the Banach space on which they operate. Then, as $t \to 0$, A_t converges weakly to A_0 and $\|A_t - A_0\|$ stays bounded. The conorm assumption means that for all small t, $\|A_t^{-1}\| \leqslant M$.

For each $v \in V$,

$$\begin{aligned} |A_t^{-1}(v) - A_0^{-1}(v)| &= |A_t^{-1} \circ (A_0 - A_t) \circ A_0^{-1}(v)| \\ &\leqslant M\, |v - A_t(A_0^{-1}(v))|. \end{aligned}$$

Since $A_0^{-1}(v)$ is fixed, and A_t converges weakly to A_0, $A_t(A_0^{-1}(v)) \to v$ as $t \to 0$, which completes the proof that A_t^{-1} converges weakly to A_0^{-1} as $t \to 0$. But also,

$$\begin{aligned} |A_t^{-1}(v) - A_0^{-1}(v)| &= |A_t^{-1} \circ (A_0 - A_t) \circ A_0^{-1}(v)| \\ &\leqslant M\, \|A_0 - A_t\|\, M\, |v| \end{aligned}$$

implies that $\|A_t^{-1} - A_0^{-1}\|$ stays bounded as $t \to 0$, and completes the proof that the inverse curve is weakly continuous. ▮

Now we return to the splitting $TM = E_t \oplus H_t$, where H_t is the hyperbolic part of the partially hyperbolic splitting for f_t, and E_t is the center part. We are assuming that $E = E_0$ is a C^1 bundle.

Let $\tilde{H}$ be a smooth approximation to H_0, and express Tf_t with respect to the splitting $TM = E \oplus \tilde{H}$ as

$$T_x f_t = \begin{bmatrix} A_{x,t} & B_{x,t} \\ C_{x,t} & K_{x,t} \end{bmatrix}.$$

Since f_t is a C^1 curve of C^2 diffeomorphisms, A, B, C, K are C^1 functions of x, t. At $t = 0$ we have

$$C_{x,0} = 0 \qquad \text{and} \qquad A_{x,0} = T_x f_0|_E$$

for all x. Furthermore, when $\tilde{H}$ closely approximates H, $\|B\|$ is small. Consequently, if $P: E \to \tilde{H}$ has norm $\leqslant 1$ then $A + BP$ is invertible and the norm of its inverse is uniformly bounded. Uniformity refers to P, x, t.

Let $\mathscr{L}$ be the vector bundle over M whose fiber at x is $\mathscr{L}_x = L(E_x, \widetilde{H}_x)$. Equipping $\mathscr{L}_x$ with the operator norm gives $\mathscr{L}$ a Finsler; let $\mathscr{L}(1)$ be its unit ball bundle. Denote by $\mathrm{Sec}(\mathscr{L})$ the Banach space of continuous sections $X: M \to \mathscr{L}$, equipped with the sup norm $\|\ \|$. Its unit ball is $\mathrm{Sec}(\mathscr{L}(1))$.

Tf_t defines a graph transform

$$\begin{tikzcd} \mathscr{L}(1) \arrow[r, "(Tf_t)_\#"] \arrow[d, "\pi"'] & \mathscr{L} \arrow[d, "\pi"] \\ M \arrow[r, "f_t"] & M \end{tikzcd}$$

according to the condition $T_x f_t(\text{graph } P) = \text{graph}((T_x f_t)_\#(P))$. That is,

$$(T_x f_t)_\# (P) = (C_{x,t} + K_{x,t} P) \circ (A_{x,t} + B_{x,t} P)^{-1},$$

which is a linear map $E_{f_t x} \to \widetilde{H}_{f_t x}$. The graph transform naturally induces a nonlinear map on the space of sections,

$$G_t: \mathrm{Sec}(\mathscr{L}(1)) \to \mathrm{Sec}(\mathscr{L})$$

such that

$$G_t(X) = (Tf_t)_\# \circ X \circ f_t^{-1}.$$

Proposition 8.4. G_t is uniformly analytic.

Remark. $(Tf_t)_\#$ is not analytic, it is only C^1. Nevertheless, for each fixed t, its action on the space of continuous sections is analytic. The uniformity refers to t.

We prove Proposition 8.4 by factoring G_t into a product of several analytic maps. Let $\mathscr{E}$, $\mathscr{E}_t$ and $\mathscr{E}_t^{-1}$ denote the bundles whose fibers at $x \in M$ are $\mathscr{E}_x = L(E_x, E_x)$, $\mathscr{E}_{x,t} = L(E_x, E_{f_t x})$, and $\mathscr{E}_{x,t}^{-1} = L(E_{f_t x}, E_x)$. Let $\mathscr{A}$, $\mathscr{A}_t$, and $\mathscr{A}_t^{-1}$ denote the invertible elements in $\mathscr{E}$, $\mathscr{E}_t$, and $\mathscr{E}_t^{-1}$, and denote sectional inversion as $\mathrm{Inv}: \mathrm{Sec}(\mathscr{A}) \to \mathrm{Sec}(\mathscr{A})$, $\mathrm{Inv}_t: \mathrm{Sec}(\mathscr{A}_t) \to \mathrm{Sec}(\mathscr{A}_t^{-1})$.

Lemma 8.5. Sectional inversion is uniformly analytic.

Proof. Consider the identity section Id of $\mathscr{A}$. Any section near Id is inverted by the power series

$$A^{-1} = \sum_{k=0}^{\infty} (\mathrm{Id} - A)^k,$$

and hence sectional inversion is analytic in a neighborhood of the identity section. For A in a neighborhood of the general section $A_0 \colon M \to \mathscr{A}$, sectional inversion factors according to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\text{Inv near } A_0"] \arrow[d, "L_{A_0^{-1}}"'] & A^{-1} \\ A_0^{-1}A \arrow[r, "\text{Inv near } Id"] & A^{-1}A_0 \arrow[u, "R_{A_0}"'] \end{tikzcd}$$

where $L_{A_0^{-1}}$ and R_{A_0} are left and right multiplication by the sections A_0^{-1} and A_0. Since $L_{A_0^{-1}}$ and R_{A_0} are continuous linear transformations of the section spaces, they are analytic, which completes the proof of the lemma for sections in a neighborhood of the identity section. The corresponding diagram

$$\begin{tikzcd} \mathrm{Sec}(\mathscr{A}_t) \arrow[r, "\text{Inv}_t \text{ near } A_0"] \arrow[d, "L_{A_0^{-1}}"'] & \mathrm{Sec}(\mathscr{A}_t^{-1}) \\ \mathrm{Sec}(\mathscr{A}) \arrow[r, "\text{Inv near } Id"] & \mathrm{Sec}(\mathscr{A}) \arrow[u, "R_{A_0}"'] \end{tikzcd}$$

applies to sectional inversion in the neighborhood of a section $A_0 \colon M \to \mathscr{A}_t$, and shows that Inv_t is analytic.

Uniform analyticity means that for any r, the rth derivative of Inv_t is uniformly bounded on sets of sections such that $\|A\|$ and $\|A^{-1}\|$ are uniformly bounded; this is clear from the higher order chain rule and the factorization of sectional inversion given above. ∎

Proof of Proposition 8.4. We have $G_t(X) = (Tf_t)_\# \circ X \circ f_t^{-1}$, and must show that G_t is a uniformly analytic function of $X \in \mathrm{Sec}(\mathscr{L})$. We factor G_t as the Cartesian product of two affine maps on section spaces, followed by inversion in one of the two spaces, followed by sectional linear composition, all of which is expressed by commutativity of

$$\begin{tikzcd} \mathrm{Sec}(\mathscr{L}) \arrow[r, "G_t"] \arrow[d, "\mathrm{Aff}_1 \times \mathrm{Aff}_2"'] & \mathrm{Sec}(\mathscr{L}) \\ \mathrm{Sec}(\mathscr{L}_t) \times \mathrm{Sec}(\mathscr{A}_t) \arrow[r, "\mathrm{Id} \times \mathrm{Inv}_t"] & \mathrm{Sec}(\mathscr{L}_t) \times \mathrm{Sec}(\mathscr{A}_t^{-1}) \arrow[u, "\text{composition}"'] \end{tikzcd}$$

where $\mathscr{L}_t$ is the bundle over M whose fiber at x is $L(E_x, \widetilde{H}_{f_t x})$, and

$$\mathrm{Aff}_1(X) = C_t + K_t X \qquad \mathrm{Aff}_2(X) = A_t + B_t X.$$

Uniform analyticity of G_t then follows from Lemma 8.5. ∎

The r^{th}-order Taylor expansion of G_t at the zero section is

$$G_t(X) = Z_t + Q_t(X) + \cdots + \frac{1}{r!}(D^r G_t)_0(X^r) + R_t(X),$$

where $Z_t = G_t(0)$, $Q_t = (DG_t)_0$.

Proposition 8.6. For small t,

(a) $t \mapsto Z_t$ is C^1.

(b) $t \mapsto (I - Q_t)^{-1}$ is weakly continuous.

(c) $\|R_t(X)\|/\|X\|^2$ is uniformly bounded for all small $X \in \mathrm{Sec}(\mathscr{L})$.

Proof. At the zero section, the 0^{th} and first derivatives of

$$G_t(X) = (C_t + K_t X)(A_t + B_t X)^{-1} \circ f_t^{-1},$$

with respect to X are computed at once as

$$Z_t = (C_t A_t^{-1}) \circ f_t^{-1}$$
$$Q_t(X) = (K_t X A_t^{-1} + C_t A_t^{-1} B_t X A_t^{-1}) \circ f_t^{-1}.$$

Since f_t is a C^1 curve of C^2 diffeomorphisms, and since the splitting $E \oplus \widetilde{H}$ is C^1, the curves $t \mapsto A_t$, $t \mapsto B_t$, $t \mapsto C_t$, $t \mapsto K_t$ in the appropriate bundles are C^1. This makes (a) immediate, and also shows that the curve $t \mapsto Q_t$ in $\mathrm{Sec}(\mathscr{L})$ is weakly continuous.

By inspection, at $t = 0$, Q_t becomes the hyperbolic operator

$$Q_0(X) = (K_0 X A_0^{-1}) \circ f_0^{-1},$$

because $C_{t=0} = 0$. Thus, for all small t, $I - Q_t$ is uniformly invertible, and Proposition 8.3 implies that $t \mapsto (I - Q_t)^{-1}$ is weakly continuous.

Assertion (c) follows from the Mean Value Theorem and the fact that the second derivative of G_t is uniformly bounded near the zero section. ∎

Proof of Theorem D. Proposition 8.6 implies that

$$G_t(X) = Z_t + Q_t(X) + R_t(X)$$

and $\|R_t(X)\| = O(1)\,\|X\|^2$ as $\|X\| \to 0$. Let $P_t\colon x \mapsto P_{x,t}$ be the unique G_t-invariant section of $\mathscr{L}$ with norm $\leqslant 1$. Thus $P_{x,t}\colon E_x \to \widetilde{H}_x$ and

$$E_{x,t} = \operatorname{graph} P_{x,t} = \{v + P_{x,t}(v) \in T_x M : v \in E_x\}.$$

Theorem D asserts that E_t is differentiable at $t=0$. That is,

$$\left.\frac{dP_{x,t}}{dt}\right|_{t=0}$$

exists and is continuous with respect to x.

Plugging $X=P_t$ into the Taylor expansion of G_t gives

$$P_t = Z_t + Q_t(P_t) + R_t(P_t),$$

and since $I-Q_t$ is invertible, we get $P_t = (I-Q_t)^{-1}\,(Z_t + R_t(P_t))$. Thus

$$\|P_t\| \leqslant \|(I-Q_t)^{-1}\|\,(\|Z_t\| + M\,\|P_t\|^2). \tag{10}$$

(These norms refer to section sup-norms or to operator norms, as appropriate.)

Now we estimate $Z_t = (C_t \circ A_t^{-1}) \circ f_t^{-1}$ as follows. It is differentiable with respect to t, and since $C_{t=0}=0$, we have $Z_{t=0}=0$. Thus $\|Z_t\| = O(1)\,t$ as $t\to 0$. Since P_t is continuous in t, and $P_0=0$, we get $\|P_t\|_0^2 \ll \|P_t\|_0$ when t is small, which lets us absorb the squared term into the l.h.s. of the inequality (10), so

$$\|P_t\| = O(1)\,t$$

as $t\to 0$. Consequently, we get a bootstrap effect on the remainder:

$$\|R_t(P_t)\| = O(1)\,t^2$$

as $t\to 0$. Combined with the more exact estimate on Z_t,

$$Z_t = tZ'_0 + o(1)\,t$$

where $Z'_0 = (d/dt)_{t=0}(Z_t)$, this gives

$$\frac{P_t}{t} = (I-Q_t)^{-1}\,Z'_0 + (I-Q_t)^{-1}\,(o(1) + O(1)\,t).$$

Proposition 8.6 implies that $(I-Q_t)^{-1}$ converges weakly to $(I-Q_0)^{-1}$ as $t\to 0$, so

$$\lim_{t\to 0}\,(I-Q_t)^{-1}\,Z'_0 = (I-Q_0)^{-1}\,Z'_0,$$

while uniform boundedness of $\|(I-Q_t)^{-1}\|$ implies that

$$\lim_{t\to 0}\,(I-Q_t)^{-1}\,(o(1)+O(1)\,t)=0.$$

Thus, as $t\to 0$,

$$\frac{P_{x,t}-P_{x,0}}{t}\to (I-Q_0)^{-1}\,Z_0',$$

uniformly in $x\in M$, which completes the proof that $t\mapsto E_t$ is differentiable at $t=0$, and that its derivative there, $(I-Q_0)^{-1}\,Z_0'$, depends continuously on $x\in M$. ∎

Remark. Suppose that E_0 and Df_t are C^r, $r\geqslant 2$. We tried to show that E_t is r^{th}-order differentiable at $t=0$ in the sense that there is an r^{th} order Taylor expansion for E_t at $t=0$. Many ingredients of the preceding proof of the $r=1$ case above generalize very nicely to $r\geqslant 2$. There is a natural notion of weak r^{th}-order differentiability, and it behaves well with respect to operator inversion and operator products. However, we would also need affirmative answers to the following two questions:

(a) Is the curve $t\mapsto(I-Q_t)^{-1}$ in $\mathrm{Sec}(\mathscr{L})$ weakly differentiable at $t=0$?

(b) Does the operator $(I-Q_0)^{-1}$ send C^1 sections of $\mathscr{L}$ to C^1 sections?

At first, it would be acceptable to assume analyticity of E_0 and f_t.

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