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# Pathological foliations and removable zero exponents

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## Introduction

The ergodic theory of uniformly hyperbolic, or “Axiom A”, diffeomorphisms has been studied extensively, beginning with the pioneering work of Anosov, Sinai, Ruelle and Bowen ([An], [Si], [Ru], [Bo]). While uniformly hyperbolic systems enjoy strong mixing properties, they are not dense among $C^1$ diffeomorphisms [Sm], [AS]. Using the concept of Lyapunov exponents, Pesin introduced a weaker form of hyperbolicity, which he termed *nonuniform hyperbolicity*. Nonuniformly hyperbolic diffeomorphisms share several mixing properties with uniformly hyperbolic ones. Our construction of the diffeomorphisms in this paper was motivated by the question of whether nonuniform hyperbolicity is dense among a large class of diffeomorphisms. As a curious by-product of our construction, we prove that a pathological feature of central foliations – the complete failure of absolute continuity – can exist in a $C^1$-open set of volume-preserving diffeomorphisms.

We recall the definition of a nonuniformly hyperbolic diffeomorphism. A real number $\lambda$ is a *Lyapunov exponent* of the diffeomorphism $g : M \to M$ if there exists a nonzero vector $v \in TM$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Tg^n(v)\| = \lambda. \tag{1}$$

By Osceledets’ Theorem [Os], if $M$ is compact, then there is a set $L \subseteq M$ which has full measure with respect to any $g$-invariant probability measure and such that the limit in (1) exists for all $v \in T_xM$ with $x \in L$. For

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a given $x \in L$, there are at most dim $(M)$ different exponents, some or all of which may be 0. The Lyapunov exponents of a uniformly hyperbolic diffeomorphism are never 0. A volume-preserving diffeomorphism is *nonuniformly hyperbolic* if the set of points in where no exponents are 0 has full volume.

Pesin [Pe] proved that if $f : M \to M$ is $C^2$ and nonuniformly hyperbolic, then $M$ may be written as the disjoint union of countably many invariant sets of positive measure on which $f$ is ergodic. He asked if nonuniform hyperbolicity is generic in $\mathrm{Diff}^r_\mu(M)$, the space of $C^r$, volume-preserving diffeomorphisms of $M$.

Pesin's question is answered in the negative for large $r$ by Cheng-Sun [CS], Herman and Xia ([He], [Xia]; see also [Yoc]). In particular, on any manifold $M$ of dimension at least 2, and for sufficiently large $r$, there are open sets of volume preserving $C^r$ diffeomorphisms of $M$ all of which posess positive measure sets of codimension one invariant tori; on each such torus, the diffeomorphism is $C^1$ conjugate to a diophantine translation. In these examples *all* of the exponents are 0 on the invariant tori.

Nonzero exponents are not confined to the uniformly hyperbolic world, however. Also germane to Pesin's question are the examples of Bonatti-Viana of volume preserving non-Axiom A diffeomorphisms all of whose exponents are nonzero ([BV]; see also [Vi] for a dissipative example). The examples are derived from Anosov diffeomorphisms through isotopy. They show that uniform hyperbolicity on most of $M$ is sometimes enough to ensure nonuniform hyperbolicity on all of $M$. These examples have the additional feature of *stability*: they lie in $C^1$-open sets of nonuniformly hyperbolic diffeomorphisms.

In this paper, we start with a diffeomorphism that is not homotopic to an Anosov diffeomorphism and which has a 0 exponent at every point. We perturb it so that the perturbations are stably nonuniformly hyperbolic. Directions on $M$ with zero exponent "borrow" some hyperbolicity from uniformly hyperbolic directions to create a new nonzero exponent that is stable. Subsequent to early versions of this paper, Dolgopyat [Do] has found other examples similar to those described here and has studied their mixing properties.

Our example relies on the theory of partially hyperbolic diffeomorphisms developed by Hirsch-Pugh-Shub [HPS], Brin-Pesin [BP], and more recently by Pugh-Shub and others ([GPS], [Wi], [PS3], [PS4]).

A further feature of these examples is that they exhibit pathological center foliations. Their holonomy maps are not absolutely continuous. Foliations exhibiting this behavior have been referred to as "Fubini's Nightmare," (also "Fubini Foiled"); Katok has previously constructed an example of a dynamically-invariant foliation with this property, which is presented in [Mi]. These center foliations show that the potential difficulties which limited Brin and Pesin in their study of partially hyperbolic diffeomorphisms and which were finally overcome in a great many cases by Grayson, Pugh and Shub do indeed exist.

For a compact $C^\infty$ manifold $M$ with volume element $\mu$, let $\mathrm{Diff}^r(M)$ be the space of $C^r$ diffeomorphisms of $M$ and let $\mathrm{Diff}^r_\mu(M)$ be the elements of $\mathrm{Diff}^r(M)$ that preserve $\mu$. Endow these spaces with the $C^r$ topology, $r \geq 1$.

Let $A_2$ be the automorphism of the 2-torus, $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let $A_3$ be the automorphism ot the 3-torus, $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$, given by $\begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}$.

**Theorem I:** *Let $\mu$ be Lebesgue measure on $\mathbf{T}^3$. Arbitrarily close to $A_3$ there is a $C^1$-open set $U \subset \mathrm{Diff}^2_\mu(\mathbf{T}^3)$ such that for each $g \in U$,*

1. *$g$ is ergodic (and a $K$-system) and nonuniformly hyperbolic.*
2. *There is an equivariant fibration $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ such that $\pi g = A_2 \pi$ The fibers of $\pi$ are the leaves of a foliation $\mathcal{W}^c_g$ of $\mathbf{T}^3$ by $C^2$ circles. In particular, the set of periodic leaves is dense in $\mathbf{T}^3$.*
3. *There exists $\lambda^c > 0$ such that, for $\mu$-almost every $w \in \mathbf{T}^3$, if $v \in T_w\mathbf{T}^3$ is tangent to the leaf of $\mathcal{W}^c_g$ containing $w$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_w g^n v\| = \lambda^c.$$

4. *Consequently, there exists a set $S \subseteq \mathbf{T}^3$ of full $\mu$-measure that meets every leaf of $\mathcal{W}^c_g$ in a set of leaf-measure 0. The foliation $\mathcal{W}^c_g$ is not absolutely continuous.*

Since a nonuniformly hyperbolic $K$-system is isomorphic to a Bernoulli shift ([Pe], Theorem 8.1), we also obtain:

**Corollary:** *The examples in Theorem I are stably Bernoulli.*

Brin, Feldman and Katok [BFK] constructed diffeomorphisms on every manifold that are Bernoulli, and Brin [Br] showed that such examples can be made to have all but one exponent not equal to zero. It is still an open question whether every manifold of dimension greater than or equal to two admits a nonuniformly hyperbolic Bernoulli diffeomorphism.

*Proof of Theorem I:* Since $A_3$ is linearly conjugate to $A_4$, the automorphism of $\mathbf{T}^3$ given by $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, it suffices to prove Theorem I for $A_4$.

By [HPS] there is a $C^1$ open set of diffeomorphisms, $U_1$, containing $A_4$ for which I.2 is satisfied. Moreover, for any $f \in U_1$ there exists a $Tf$-invariant continuous splitting of the tangent bundle to $\mathbf{T}^3$ into 3 subbundles

$$T\mathbf{T}^3 = E^u_f \oplus E^c_f \oplus E^s_f,$$

with $E^u_f$, $E^c_f$ and $E^s_f$ all 1-dimensional. There are Finslers such that $E^u_f$ is expanded by $Tf$, $E^s_f$ is contracted by $Tf$ and the expansion and contraction of $Tf$ on $E^c_f$ is in between. That is, for all $w \in \mathbf{T}^3$ and unit vectors $v^u \in E^u_f(w)$, $v^c \in E^c_f(w)$ and $v^s \in E^s_f(w)$,

$$\begin{aligned} r^u_f(w) &= \|T_w f(v^u)\| > 1, \\ r^s_f(w) &= \|T_w f(v^s)\| < 1, \text{ and} \\ r^c_f(w) &= \|T_w f(v^c)\|, \quad \text{where} \quad r^s_f(w) < r^c_f(w) < r^u_f(w). \end{aligned}$$

Moreover, the splitting

$$T\mathbf{T}^3 = E^u_f \oplus E^c_f \oplus E^s_f,$$

varies continuously with $f$, as do the functions $r^u_f$, $r^c_f$ and $r^s_f$. For a volume-preserving ergodic $f$ in $U_1$, there are three Lyapunov exponents, which by Osceledets' theorem may be expressed as integrals:

$$\begin{aligned} \lambda^u(f) &= \int_{\mathbf{T}^3} \log r^u_f(w)\, d\mu(w) \\ \lambda^c(f) &= \int_{\mathbf{T}^3} \log r^c_f(w)\, d\mu(w) \\ \lambda^s(f) &= \int_{\mathbf{T}^3} \log r^s_f(w)\, d\mu(w) \end{aligned}$$

Regardless of whether $f$ is ergodic, the functions $\lambda^\xi(f)$, for $\xi = u, c, s$, are still defined; they are the average Lyapunov exponents for $f$ and depend continuously on $f$. While $r^u, r^c$, and $r^s$ depend on the choice of Finsler, the functions $\lambda^u, \lambda^c$, and $\lambda^s$ do not. Note that $\lambda^c(A_4) = 0$ and $\lambda^u(A_4) = -\lambda^s(A_4) = \log(m)$, where $m = (3+\sqrt{5})/2$.

We will find a stably ergodic volume preserving $f$ in $U_1$ arbitarily close to $A_4$, for which $\lambda^c(f) > 0$. This will complete the proof. For, let $U_2 \subset \mathrm{Diff}^2_\mu(\mathbf{T}^3)$ be a $C^1$ neighborhood of $f$ contained in $U_1$ consisting of ergodic diffeomorphisms and chosen small enough that $\lambda^c(g)$ is small and positive for all $g \in U_2$. Then I.1, I.2, I.3 hold for all $g \in U_2$ by construction. Let $S$ be the full measure set of points where $\lambda^c(g)$ exists. If any circle leaf of $\mathcal{W}^c_g$ intersected $S$ in a set of positive measure, then that circle leaf would increase exponentially in length under iterates of $g$. But these lengths are bounded by I.2, and so I.4 follows.

It remains only to find a stably ergodic volume preserving $f$ in $U_1$, arbitarily close to $A_4$, for which $\lambda^c(f) > 0$. We find such $f$ in a 2-parameter family of diffeomorphisms $f_{a,b}$ which we now describe.

Let $\psi : \mathbf{T} \to \mathbf{T}$ be any nonconstant null-homotopic $C^3$ function, let $v_0$ be an eigenvector for $A_2$ corresponding to the eigenvalue $m = (3+\sqrt{5})/2$, and let $\varphi(x, y) = \sin(2\pi y)$. For $a, b \in \mathbf{R}$, let

$$f_{a,b} = g_a \circ h_b,$$

where

$$h_b(x, y, z) = (2x + y, x + y, z + x + y + b\varphi(x, y)), \text{ and}$$
$$g_a(x, y, z) = (x, y, z) + (a\psi(z)v_0, 0).$$

The diffeomorphisms $f_{0,b} = h_b$ are skew products and for $b \neq 0$ they are stably ergodic (and in fact, stably $K$-systems). By Corollary B1 in [BW] they are stably $K$-systems if they are ergodic among skew products, and by [AKS] they are ergodic among skew products for all $b \neq 0$. (This also follows directly from the arguments in [BW]). Thus, $f_{a,b}$ is ergodic for a sufficiently small $a$ depending on $b$. Writing $\lambda^\xi(a, b) = \lambda^\xi(f_{a,b})$, $\xi = u, c, s$, we prove in Lemma 1.2 that $\lambda^u(a, b) + \lambda^c(a, b)$ and $-\lambda^s(a, b)$ are constant and equal to $\log(m)$.

In Section 1 we prove:

**Proposition II:** *The function $\lambda^u$ is $C^2$ in a neighborhood of $(0, 0)$. Further,*

$$\frac{\partial}{\partial a}\lambda^u(0, b) = 0, \quad \textit{and}$$
$$\frac{\partial^2}{\partial a^2}\lambda^u(0, 0) = -u_0^2 \int_0^1 \psi'(z)^2\, dz < 0,$$

*where $u_0 = ((1, 1) \cdot v_0)/(m - 1)$.*

Thus for arbitrarily small and positive $a, b$ we have $f_{a,b}$, volume preserving, stably ergodic and $\lambda^c(f_{a,b}) > 0$. □

Our proof gives some hope that a variant of Pesin's original question holds true for volume-preserving diffeomorphisms: either all exponents are zero ($\mu$-a.e.), or, as with our examples, the system may be perturbed to become stably nonuniformly hyperbolic.

**Question 1a):** *For $r \geq 1$, is it true for generic $f$ in $\mathit{Diff}^r_\mu(M)$ that for almost every ergodic component of $f$, either all of the Lyapunov exponents of $f$ are 0 or none are 0 ($\mu$-a.e.)?*

A special case of 1a) is 1b).

**Question 1b):** *For $r \geq 1$ does the generic ergodic diffeomorphism in $\mathit{Diff}^r_\mu(M)$ have either no exponent equal to 0 or all exponents equal to 0 ($\mu$-a.e.)?*

Question 1a) has an affirmative answer for 2-dimensional $M$ in the case $r = 1$; Mañé has shown that the generic diffeomorphism in $\mathrm{Diff}^1_\mu(M)$ either has all of its Lyapunov exponents zero or is an Anosov diffeomorphism ([Ma1], [Ma2]).

An analogue of Question 1 for $\mathrm{Diff}^r(M)$ is the following.

**Question 2:** *For $r \geq 1$, is it true for the generic $f$ in Diff$^r(M)$ and any weak limit $\nu$ of averages of the push forwards $\frac{1}{n}\sum_1^n f_*^j \mu$ that almost every ergodic component of $\nu$ has some exponents not equal to 0 ($\nu$-a.e.)? All exponents not equal to 0?*

Question 1b) is closest in spirit to Theorem I, which in turn gives some credence to the possibility that 1a) is true.

One of the achievements of the theory of uniformly hyperbolic dynamical systems were the theorems of Sinai, Ruelle and Bowen on invariant measures on the attractors of a system. These attractors and measures are now called Sinai-Ruelle-Bowen measures and SRB measures (or SRB attractors), for short. They may also be called ergodic attractors.[1]

Given $f \in \mathrm{Diff}^r(M)$ (not necessarily preserving $\mu$), a closed, $f$-invariant set $A \subset M$ and an $f$-invariant ergodic measure $\nu$ on $A$, we define $B(A, \nu)$, the *basin* of $A$, to be the set of points $x \in M$ such that $f^n(x) \to A$ and for every continuous function $\phi : M \to \mathbf{R}$

$$\lim_{n\to\infty} \frac{1}{n}(\phi(x) + \cdots + \phi(f^n(x))) \to \int_A \phi(x)\, d\nu.$$

**Definition:** $\nu$ is an *SRB measure* and $A$ is an *SRB (or ergodic) attractor* if the Lebesgue measure of $B(A, \nu)$ is positive.

It follows from the definition that a diffeomorphism has at most countably many SRB measures. Sinai, Ruelle and Bowen proved that for $r \geq 2$ and $f$ an Axiom A, no-cycle diffeomorphism (see [Si], [Ru], [Bo]), almost every point in $M$ with respect to Lebesgue measure $\mu$ is in the basin of an SRB measure, and there are only finitely many SRB measures.

**Question 3:** *For $r \geq 2$, is it true for generic $f$ in Diff$^r(M)$ that the union of the basins of the SRB attractors of $f$ has full Lebesgue measure in $M$?*

This natural question is on the minds of quite a few people. See [PS2], [Pa], [BV] for discussions and examples. Question 2) might be a way to approach Question 3) along the lines of [Pe], [PS1], [PS4].

We remark here that our construction can be slightly modified to obtain diffeomorphisms of $\mathbf{T}^n$, for any $n \geq 3$, that satisfy the conclusions of Theorem I and its corollary. In this modification, the automorphism $A_2$ of $\mathbf{T}^2$ is replaced by an Anosov automorphism of $\mathbf{T}^{n-1}$ with one-dimensional expanding eigenspace.

We thank Michael Herman for many conversations which clarified Questions 1–3 for us. Some of this material was presented and discussed in his

[1] We take some of the conclusions of the theorems of Sinai, Ruelle, and Bowen as a definition and warn the reader that the use of SRB measure or attractor is not uniform in the literature. For a survey of SRB measures (using a different definition) see [You2].

seminar during March 1998. The question of whether perturbing a skew product diffeomorphism over an Anosov could produce nonzero exponents was raised by Lai-Sang Young during a conversation about these questions. We thank her for reminding us that examples such as the ones we construct might exist. Numerical experiments conducted by Chai Wah Wu and later by Niels Sondergaard convinced us of their existence and we are indebted to them. We also thank Charles Pugh and Clark Robinson for useful conversations and the referee for pointing out several references to us.

## 1. Behavior of exponents inside the family $f_{a,b}$

Let $\mathcal{A}$ be a neighborhood of the origin in $\mathbf{R}^2$ such that $f_\omega$ satisfies conclusion 2 of Theorem I for all $\omega \in \mathcal{A}$. (Recall that the existence of $\mathcal{A}$ is ensured by Corollary 8.3 in [HPS]). Let $\lambda^u$, $\lambda^c$ and $\lambda^s$ be defined as in the previous section. For $\xi = u, c, s$, and $\omega \in \mathcal{A}$, let $E^\xi_\omega = E^\xi_{f_\omega}$. Note that the center-unstable distribution $E^u_\omega \oplus E^c_\omega$ does not depend on $\omega \in \mathcal{A}$. It is the constant distribution spanned by the vector fields $\partial/\partial z$ and $v_0 \cdot (\partial/\partial x, \partial/\partial y)$, which we will denote by $E^{uc}$. This implies that $\lambda^u + \lambda^c$ is the constant function on $\mathcal{A}$, as the next lemma and the argument that follows makes precise.

**Lemma 1.1** *There exists a $C^\infty$ 2-form $\alpha$ such that $\alpha$ is nondegenerate on $E^{uc}$, and for all $\omega \in \mathcal{A}$,*

$$f^*_\omega \alpha = m\alpha + \beta_\omega,$$

*where $\beta_\omega$ vanishes on $E^{uc}$:*

$$\beta_\omega(v_1, v_2) = 0, \qquad \forall v_1, v_2 \in E^{uc}.$$

*Proof of Lemma 1.1:* Write $v_0 = (q_1, q_2)$ and let $\alpha = v_0 \cdot (dx, dy) \wedge dz = q_1 dx \wedge dz + q_2 dy \wedge dz$. □

Using $\alpha$ and the volume form $d\mu$, we may now choose for each $\omega$ a Finsler (in fact, a continuous Riemann structure) on $\mathbf{T}^3$ so that for $\xi = u, c, s$, the functions $r^\xi_\omega = r^\xi_{f_\omega}$ satisfy the equations:

$$r^u_\omega(w) r^c_\omega(w) r^s_\omega(w) = 1, \quad \text{and} \tag{2}$$
$$r^u_\omega(w) r^c_\omega(w) = m, \tag{3}$$

for all $w \in \mathbf{T}^3$. To accomplish this, we shall choose at each point $w \in \mathbf{T}^3$ an appropriate orthonormal basis $v^u(w), v^c(w), v^s(w)$.

For $\omega \in \mathcal{A}$, the line $E^u_\omega(w)$ sits inside the plane $E^{uc}(w)$, transverse to the line spanned by $(0, 0, 1)$. Let $v^u(w)$ be the unique vector in $E^u_\omega(w)$ of the form $(v_0, u)$, for some $u \in \mathbf{R}$. Choose $v^c(w) \in E^c_\omega(w)$ and $v^s(w) \in E^s_\omega(w)$ so that $\alpha(v^u(w), v^c(w)) = 1$ and $d\mu(v^u(w), v^c(w), v^s(w))$. Lemma 1.1 now gives (2), and (3) follows from $f^*_\omega d\mu = d\mu$. From the definitions of $\lambda^\xi(\omega)$, we immediately obtain:

**Lemma 1.2** *The functions $\lambda^s$ and $\lambda^u + \lambda^c$ are constant:*

$$\lambda^s(\omega) = -\log(m), \qquad \lambda^u(\omega) + \lambda^c(\omega) = \log(m).$$

*for $\omega \in \mathcal{A}$.*

Our analysis of $\lambda^u$, $\lambda^c$, $\lambda^s$ is thus reduced to a study of $\lambda^u$.

### *1.1. Proof of Proposition II*

The proof of Proposition II breaks into three parts. In Lemma 1.5, we show that $\lambda^u(\omega)$ is a $C^2$ function of $\omega \in \mathcal{A}$. In Lemma 1.6 we show that the partial derivative $\partial\lambda^u/\partial a$ is 0 at $a = 0$, and in Lemma 1.7 we show that $\partial^2\lambda^u/\partial a^2$ is negative at $(0, 0)$.

As remarked above, for $\omega \in \mathcal{A}$ and $w \in \mathbf{T}^3$, there is a unique vector in $E^u_\omega(w)$ of the form $v^u(w) = (v_0, u_\omega(w))$. This defines a continuous function $u_\omega : \mathbf{T}^3 \to \mathbf{R}$.

**Lemma 1.3** *For $\omega = (a, b) \in \mathcal{A}$,*

$$\lambda^u(\omega) = \log(m) - \int_{\mathbf{T}^3} \log(1 - a\psi'(w)u_\omega(w))\, d\mu(w).$$

*Proof of Lemma 1.3:* Let $\varphi_b(x, y, z) = x + y + b\varphi(x, y)$. From the definition of $f_\omega$, we have:

$$T_w f_\omega \begin{pmatrix} v_0 \\ u_\omega(w) \end{pmatrix} = \begin{pmatrix} \left[m + a\psi'(f_\omega(w))\left[u_\omega(w) + \nabla\varphi_b(w)\cdot v_0\right]\right] v_0 \\ u_\omega(w) + \nabla\varphi_b(w)\cdot v_0 \end{pmatrix}$$

$$= \begin{pmatrix} \left[m + a\psi'(f_\omega(w))r^u_\omega(w)u_\omega(f_\omega(w))\right] v_0 \\ r^u_\omega(w)u_\omega(f_\omega(w)) \end{pmatrix}.$$

It follows that

$$r^u_\omega(w) = m + a\psi'(f_\omega(w))u_\omega(f_\omega(w))r^u_\omega(w),$$

and so $r^u_\omega(w) = m/(1 - a\psi'(f_\omega(w))u_\omega(f_\omega(w)))$. To obtain the formula, integrate $\log(r^u_\omega(w))$. □

We next establish the smoothness of $u_\omega$.

**Lemma 1.4** *There exists a neighborhood $\mathcal{A}$ of $(0, 0)$ in $\mathbf{R}^2$, such that, for each $w \in \mathbf{T}^3$, the function $\omega \mapsto u_\omega(w)$ is $C^2$ on $\mathcal{A}$. The first two derivatives of this function depend uniformly on $w$.*

*Proof of Lemma 1.4:* It does not affect the smoothness of $u_\omega$ if we scale the functions $\varphi$ and $\psi$ by a positive constant. Thus we may assume that

$$(\|\psi\|_0\|v_0\| + \|\varphi\|_0 + 1)^2 < m. \tag{4}$$

Let $\mathcal{W}^{uc}$ be the $C^\infty$ foliation tangent to $E^{uc}$. The leaves of $\mathcal{W}^{uc}$ are smoothly permuted by $f_\omega$. Let $X$ be the disjoint union of the leaves of $\mathcal{W}^{uc}$. Because the foliation structure of $\mathcal{W}^{uc}$ is preserved by all of the $f_\omega$, there is a well-defined, $C^3$ map $\mathcal{F} : \mathcal{A}_0 \times X \to \mathcal{A}_0 \times X$ given by:

$$\mathcal{F}(\omega, w) = (\omega, f_\omega(w)).$$

On $\mathcal{A}_0 \times X$, put the metric:

$$d((\omega_1, w_1), (\omega_2, w_2)) = \max\{d_{\mathcal{A}_0}(\omega_1, \omega_2), d_X(w_1, w_2)\},$$

where $d_X$ is the induced Riemannian metric on $X$ and $d_{\mathcal{A}_0}((a_1, b_1), (a_2, b_2)) = \max\{|a_1 - a_2|, |b_1 - b_2|\}$. With respect to this metric, there exists a constant $\rho$ such that

$$d(\mathcal{F}(\omega_1, w_1), \mathcal{F}(\omega_2, w_2)) \geq \rho d((\omega_1, w_1), (\omega_2, w_2)),$$

for all $\omega_1, \omega_2 \in \mathcal{A}_0$ and $w_1, w_2 \in X$. The constant $\rho$ is the inverse of the Lipschitz norm of $\mathcal{F}^{-1}$. A straightforward estimate shows that by shrinking the size of the neighborhood $\mathcal{A}_0$, we may bring $\rho$ arbitrarily close to:

$$(\|\psi\|_0\|v_0\| + \|\varphi\|_0 + 1)^{-1}.$$

Let $\mathcal{B}$ be the trivial bundle over $\mathcal{A}_0 \times X$ whose fiber $\mathcal{L}_w = \mathcal{L}$ over $w$ is the set of all linear maps $L : E^u_{\omega_0}(w) \to E^c_{\omega_0}(w)$. Since $E^u_{\omega_0}$ and $E^c_{\omega_0}$ are 1-dimensional, so is $\mathcal{L}$. We think of $\mathcal{B}$ as the product $\mathcal{A}_0 \times X \times \mathbf{R}$.

With respect to the $C^\infty$ splitting $TX = E^{uc} = E^u_{\omega_0} \oplus E^c_{\omega_0}$, the map $Tf_\omega|_{E^{uc}}$ can be written:

$$Tf_\omega = \begin{pmatrix} A_\omega & B_\omega \\ C_\omega & K_\omega \end{pmatrix},$$

where $A_\omega : E^u_{\omega_0} \to E^u_{\omega_0}$, $B_\omega : E^c_{\omega_0} \to E^u_{\omega_0}$, $C_\omega : E^u_{\omega_0} \to E^c_{\omega_0}$, and $K_\omega : E^c_{\omega_0} \to E^c_{\omega_0}$ and $\omega_0 = (0, 0)$. These maps depend in a $C^2$ fashion on $\omega$ and on the basepoint $w \in \mathbf{T}^3$. When $\omega = \omega_0$, we have $B = C = 0$, $K = 1$, and $A = m$.

Define a bundle map $\mathcal{F}_\sharp : \mathcal{B} \to \mathcal{B}$, covering $\mathcal{F}$, by:

$$\mathcal{F}_\sharp(\omega, w, L) = (\omega, f_\omega(w), (C_\omega(w) + K_\omega(w)L)(A_\omega(w) + B_\omega(w)L)^{-1}).$$

Then $\mathcal{F}_\sharp$ is $C^2$, contracts fibers of $\mathcal{B}$ at the weakest by a factor $\sigma \doteq m^{-1}$, and has strongest base contraction by the factor $\rho \doteq (\|\psi\|_0\|v_0\| + \|\varphi\| + 1)^{-1}$. These estimates depend uniformly on the size of the neighborhood $\mathcal{A}_0$. Thus, by inequality (4), there is a neighborhood $\mathcal{A} \subseteq \mathcal{A}_0$ of $\omega_0$, in which

$$\sigma\rho^{-2} < 1.$$

By the $C^r$ Section Theorem of [HPS] (see also [Sh]), the unique $\mathcal{F}_\sharp$-invariant section $s : \mathcal{A} \times X \to \mathcal{L}$ is $C^2$. But the graph of $s(\omega, w) : E^u_{\omega_0}(w) \to E^s_{\omega_0}(w)$ is precisely $E^u_\omega(w)$. We conclude that $E^u_\omega(w)$, and thus $u_\omega(w)$, is a $C^2$ function of $\omega \in \mathcal{A}$ and of $w \in X$. Since $\mathcal{W}^{uc}$ is a $C^\infty$ foliation, these estimates are uniform over $w \in \mathbf{T}^3$. □

**Lemma 1.5** *$\lambda^u$ is $C^2$ on $\mathcal{A}$, and*

$$\frac{\partial}{\partial a}\lambda^u(a,b) = \int_{\mathbf{T}^3} \frac{\psi'(w)u_{a,b}(w) + a\psi'(w)\frac{\partial u_{a,b}(w)}{\partial a}}{1 - a\psi'(w)u_{a,b}(w)} d\mu(w). \tag{5}$$

*Proof:* By Lemma 1.4, the function $\omega \mapsto u_\omega(w)$ is $C^2$ on $\mathcal{A}$, uniformly in $w$. Then by the formula in Lemma 1.3, $\lambda^u$ is $C^2$ as well. Differentiating this formula with respect to $a$ gives (5). □

Setting $a = 0$ in (5), we obtain:

$$\frac{\partial}{\partial a}\lambda^u(0,b) = \int_{\mathbf{T}^3} \psi'(w)u_{0,b}(w)\, d\mu(w) \tag{6}$$

The distribution $E^u_{0,b}$ for the skew product $f_{0,b}$ is invariant under translations of the form $(x, y, z) \mapsto (x, y, z + z_0)$. This implies that the function $u_{0,b}(x, y, z)$ depends only on $x$ and $y$. On the other hand, $\psi'(x, y, z) = \psi'(z)$ depends only on $z$. The integral in (6) is therefore equal to

$$\int_{\mathbf{T}} \psi'(z)\, dz \int_{\mathbf{T^3}} u_{0,b}(w)\, d\mu(w) = 0,$$

since $\psi$ is homotopic to a constant map. We have shown:

**Lemma 1.6** *For $(a, b) \in \mathcal{A}$,*

$$\frac{\partial}{\partial a}\lambda^u(0,b) = 0$$

The behavior of the exponents of $f_{a,b}$ near $(0, 0)$ is thus determined by the second derivative of $\lambda^u$ with respect to $a$. An exact computation of this second derivative is difficult in general. For our purposes, it suffices to compute this derivative at $(0, 0)$.

**Lemma 1.7**

$$\frac{\partial^2}{\partial a^2}\lambda^u(0,0) = -u_0^2 \int_0^1 \psi'(z)^2\, dz < 0,$$

*where $u_0 = ((1, 1) \cdot v_0)/(m - 1)$.*

*Proof of Lemma 1.7:* Use "$f_a$" to denote $f_{a,0}$, "$u_a$" to denote $u_{a,0}$, and "$\lambda(a)$" for $\lambda^u(a, 0)$. Differentiating (5), we have

$$\lambda''(a) = \int_{\mathbf{T}^3} \left( \frac{\psi'(w)u_a(w) + a\psi'(w)\frac{\partial u_a(w)}{\partial a}}{1 - a\psi'(w)u_a(w)} \right)^2 + \frac{2\psi'(w)\frac{\partial u_a(w)}{\partial a} + a\psi'(w)\frac{\partial^2 u_a(w)}{\partial a^2}}{(1 - a\psi'(w)u_a(w))} \, d\mu(w),$$

and setting $a = 0$, we obtain

$$\lambda''(0) = \int_{\mathbf{T}^3} (\psi'(w)u_0(w))^2 + 2\psi'(w)\frac{\partial u_a(w)}{\partial a}|_{a=0} \, d\mu(w). \tag{7}$$

The map $A_4 = f_{0,0}$ is linear. It is easy to see that $u_0(w)$ is the constant function $u_0(w) = u_0 = ((1, 1) \cdot v_0)/(m - 1) \neq 0$

For $a \in \mathbf{R}$, $w = (x, y, z) \in \mathbf{T}^3$, and $u \in \mathbf{R}$, let

$$\gamma(a, w, u) = \frac{c + u}{m + a\psi'(w)(c + u)},$$

where $c = (1, 1) \cdot v_0$. Note that $\gamma(a, w, u_a(f_a^{-1}(w))) = u_a(w)$ and that for $|a|$ sufficiently small,

$$u_a(w) = \lim_{n\to\infty} \gamma(a, w, \gamma(a, f_a^{-1}(w), \ldots \gamma(a, f_a^{-n}(w), 0) \cdots)).$$

We compute:

$$\frac{\partial \gamma}{\partial a}(a, w, u) = \frac{-\psi'(w)(c + u)^2}{(m + a\psi'(w)(c + u))^2},$$

and set $a = 0$ to obtain:

$$\frac{\partial \gamma}{\partial a}(0, w, u) = \frac{-\psi'(w)(c + u)^2}{m^2}.$$

Similarly,

$$\frac{\partial \gamma}{\partial x}(0, w, u) = \frac{\partial \gamma}{\partial z}(0, w, u) = \frac{\partial \gamma}{\partial y}(0, w, u) = 0,$$

and

$$\frac{\partial \gamma}{\partial u}(0, w, u) = \frac{1}{m}.$$

We want to evaluate $\frac{\partial u_a(w)}{\partial a}|_{a=0}$. Since $u_a(w) = \gamma(a, w, u_a(f_a^{-1}(w)))$, the chain rule yields:

$$\begin{aligned}\frac{\partial u_a(w)}{\partial a}|_{a=0} &= \frac{\partial}{\partial a}\gamma(a, w, u_a(f_a^{-1}(u)))|_{a=0}\\ &= \frac{\partial \gamma}{\partial a}(0, w, u_0(f_0^{-1}(w)))\\ &\quad + \frac{\partial \gamma}{\partial u}(0, w, u_0(f_0^{-1}(w))) \cdot \frac{\partial u_a(f_a^{-1}(w))}{\partial a}|_{a=0})\\ &= \frac{-\psi'(w)(c + u_0(f_0^{-1}(w)))^2}{m^2} + \frac{1}{m}\frac{\partial u_a(f_a^{-1}(w))}{\partial a}|_{a=0}).\end{aligned}$$

Recall that $u_0$ is the constant function $u_0(w) = u_0$, and $u_0 = (c + u_0)/m$, so this expression simplifies to:

$$\frac{\partial u_a(w)}{\partial a}|_{a=0} = -\psi'(w)u_0^2 + \frac{1}{m}\frac{\partial u_a(f_a^{-1}(w))}{\partial a}|_{a=0}. \tag{8}$$

Iterating (8) gives

$$\frac{\partial u_a(w)}{\partial a}|_{a=0} = -\psi'(w)u_0^2 - \frac{\psi'(w_{-1})u_0^2}{m} - \frac{\psi'(w_{-2})u_0^2}{m^2} - \cdots$$

where $w_{-j} = f_a^{-j}(w)$. Hence

$$\int_{\mathbf{T}^3} \psi'(w)\frac{\partial u_a(w)}{\partial a}|_{a=0}\, d\mu(w) = -\sum_{j\geq 0}\int_{\mathbf{T}^3} \frac{\psi'(w)\psi'(w_{-j})u_0^2}{m^j}\, d\mu(w).$$

The $j = 0$ term of this sum is $-\int \psi'(w)^2 u_0^2\, d\mu(w)$. Pulling out this term, we have

$$\int_{\mathbf{T}^3} (\psi'(w)u_0(w))^2 + 2\psi'(w)\frac{\partial u_a(w)}{\partial a}|_{a=0}\, d\mu(w) =$$

$$= -\int_{\mathbf{T}^3} \psi'(w)^2 u_0^2\, d\mu(w) - 2\sum_{j\geq 1}\int_{\mathbf{T}^3} \frac{\psi'(w)\psi'(w_{-j})u_0^2}{m^j}\, d\mu(w). \tag{9}$$

Notice that

$$\begin{aligned}&\int_{\mathbf{T}^3} \psi'(w)\psi'(w_{-j})u_0^2\, d\mu(w) =\\ &\qquad = u_0^2 \int_0^1\int_0^1\int_0^1 \psi'(z)\psi'(z - r_j x - s_j y)\, dx\, dy\, dz,\end{aligned}$$

where $r_j$ and $s_j$ are integers, and $r_j = 0$ if and only if $j = 0$, in which case $s_j = 0$ as well. Thus, for $j \geq 1$,

$$\begin{aligned}
\int_{\mathbf{T}^3} \psi'(w)\psi'(w_{-j})u_0^2\, d\mu(w) = \\
&= u_0^2 r_j^{-1} \int_0^1 \int_0^1 \psi'(z)\psi(z - r_j x - s_j y))|_{x=0}^{x=1}\, dy\, dz \\
&= 0.
\end{aligned}$$

Combining this with equations (7) and (9), we have:

$$\begin{aligned}
\lambda''(0) &= -\int_{\mathbf{T}^3} \psi'(w)^2 u_0^2\, d\mu(w) - 2\sum_{j\geq 1} \int_{\mathbf{T}^3} \frac{\psi'(w)\psi'(w_{-j})u_0^2}{m^j}\, d\mu(w) \\
&= -u_0^2 \int_0^1 \psi'(z)^2\, dz,
\end{aligned}$$

completing the proof.

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