Contemporary Mathematics
Volume 64, 1987

SOME LINEARLY INDUCED MORSE-SMALE SYSTEMS, THE QR ALGORITHM AND THE TODA LATTICE

Michael Shub and Alphonse Thomas Vasquez

There have recently been papers "explaining" the QR algorithm and linking it to the Toda lattice. Here we present a dynamical systems perspective which is simple yet more structured and general. We thank Etienne Ghys, John Smillie, Dick Sacksteder and the students in Math. U719 for useful conversations.

Given a group G, a subgroup U of G, and an element $g \in G$, attempt to conjugate g into U; that is attempt to find an element $h \in G$ such that $h^{-1}gh \in U$. Groups G of special interest are GL(n), the n × n invertible matrices over the reals or complexes, and U, the subgroup of upper triangular matrices (the eigenvalues of an upper triangular matrix are its diagonal entries). In the attempt to conjugate g into U it is quite natural to consider the dynamics of g on G/U. For if we find a fixed coset hU, that is, $ghU = hU$, then it follows that $h^{-1}ghU = U$ and that $h^{-1}gh \in U$. So the $h \in G$ such that $h^{-1}gh \in U$ correspond to the fixed points, hU, of the action of g on G/U. Denote this action by $\varphi_g: G/U \to G/U$.

A main result of this paper is:

Theorem 1. *Suppose that the eigenvalues of $g \in$ GL(n) have distinct moduli, then $\varphi_g: G/U \to G/U$ is a Morse-Smale diffeomorphism. It has n! fixed points and no periodic points other than these. The stable manifold of one of the fixed points is an open, dense set.*

The introduction of the space X = G/U was strongly suggested by D. S. Watkins' SIAM review article [Wa]. He rightly emphasized

---

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F, 65F.

the fact that the QR-process is "simultaneous iteration" and as such is a natural extension of the power method. It seems to us that the space X is the natural mathematical object to invoke to discuss "simultaneous subspace iteration." This is because $aU = bU \in X$ if and only if for $1 \leq i \leq n$ the vector subspace spanned by the first i columns of a equals the vector subspace spanned by the first i columns of b.

A diffeomorphism, $\varphi : M \overset{\cong}{\to} M$, is *Morse-Smale* if and only if:

(i) The set, $\mathrm{Per}(\varphi)$, of periodic points is finite.

(ii) The periodic points of φ are hyperbolic; i.e. for each $x \in \mathrm{Per}(\varphi)$, $T_x(\varphi^n)$: $T_x(M) \to T_x(M)$ has no eigenvalue of modulus 1; here n is the *period* of $x \equiv$ the least positive integer $k \ni : \varphi^k(x) = x$.

(iii) $$M = \bigcup_{x \in \mathrm{Per}(\varphi)} W^s(x) = \bigcup_{x \in \mathrm{Per}(\varphi)} W^u(x)$$

where (with n denoting the period of x):

$W^s(x)$ = the *stable manifold* of $x = \{y \in M \mid \lim_{k\to\infty} \varphi^{kn}(y) = x\}$

and

$W^u(x)$ = the *unstable manifold* of $x = \{y \in M \mid \lim_{k\to-\infty} \varphi^{kn}(y) = x\}$.

(iv) $W^s(x_1)$ and $W^u(x_2)$ are submanifolds meeting transversally for any $x_\alpha \in \mathrm{Per}(\varphi)$. See Smale [Sm] for more details.

Remarks. The theorem implies: *for any* $x \in X$, $\{\varphi_g^m(x)\}$ *converges to a fixed point of* φ_g. *Furthermore, for almost all* x, *the limit is the same fixed point.*

If g is a diagonalizable matrix with multiple eigenvalues but no eigenvalues of equal modulus apart from multiple eigenvalues it is still true and fairly simple (see e.g. Lemma 4 and the material immediately thereafter) to prove that φ_g is linearly globally-convergent in the sense that $\varphi_g^k(x)$ converges linearly to a fixed point for any $x \in G/U$.

Let $G = K \times U'$ and $\pi: G \to K$ be the projection onto K induced by this product decomposition. For $g \in G$, left multiplication by g on G, $L_g: G \to G$ induces a map $K_g: K \to K$ by the formula: $K_g(k) = \pi(gk)$. The diagram

$$\begin{tikzcd} G \arrow[r, "L_g"'] \arrow[d, "\pi"] & G \arrow[d, "\pi"] \\ K \arrow[r, "K_g"'] & K \end{tikzcd}$$

commutes, as can readily be seen by identifying K with the coset space G/U'. It also follows that a 1-parameter group, g_t, acting on the left of G induces a flow, $\phi_t = K_{g_t}$ on K, i.e. $\phi_{s+t} = \phi_s \phi_t$.

In the case of GL(n) over the reals and complexes respectively take K to be the orthogonal and unitary subgroups, O(n) and U(n) respectively, and U' to be the subgroup of upper triangular matrices with real positive entries on the diagonal. The product formula GL(n) = $K \times U'$ is closely related to the Iwasawa decomposition of GL(n) while the factorization, A = QR, for an individual matrix A is called the QR factorization of A.

If U' is a subgroup of U, which is true in the cases we are discussing, then the following diagram

$$\begin{tikzcd} G \arrow[r, "L_g"'] \arrow[d, "\pi"] & G \arrow[d, "\pi"] \\ K \arrow[r, "K_g"'] \arrow[d] & K \arrow[d] \\ G/U \arrow[r, "\varphi_g"'] & G/U \end{tikzcd}$$

commutes where once again K is identified with G/U'. The fiber of $K \to G/U$ is identified with U/U'. So in the real case U/U' is finite and in the complex case U/U' is a product of n circles, $S^1 \times \cdots \times S^1$.

Corollary 1. Suppose that the eigenvalues of $g \in GL(n)$ have distinct moduli. Then for any element $k \in K$, $\{K_g^m(k)\}$ converges to an invariant fiber of $K \to G/U$ (in fact $K_g^m(k)$ is forward asymptotic with $K_g^m(h)$ for a particular h in the fiber) and $K_g^m(k)^{-1} g K_g^m(k)$ "tends to" the upper triangular matrices. If g lies on a one parameter group g_t and $\phi_t = K_{g_t}$ then the same is true for $\phi_t(k)$ and $\phi_t(k)^{-1} g \phi_t(k)$.

Remark. The parenthetical remark of the corollary follows since the fiber is a normally hyperbolic invariant manifold [HPS]. Consequently the diagonal elements tend to constants, the below diagonal elements tend to zero, and the above diagonal elements tend to rotate by the difference of the arguments of the eigenvalues [Wi-1966].

Except for the parenthetical remark here is an elementary proof. Identify $X = G/U$ with K/K' where $K' = K \cap U$. Let $q_0 \in K$ be such that $\lim_{m\to\infty} \varphi_g^m(kU) = q_0 U$. Since this limit is obviously fixed by φ_g, $q_0^{-1} g q_0 \in U$. The topology of the fibration $K \to K/K'$ implies that there is a sequence, $k_m \in K'$, such that $\lim_{m\to\infty} K_g^m(k) k_m = q_0 \in K$. Thus $\lim_{m\to\infty} k_m^{-1} (K_g^m(k)^{-1} g K_g^m(k)) k_m \in U$. Since the k_m's are diagonal matrices with diagonal entries of modulus 1, it follows that the diagonal entries of $K_g^m(k)^{-1} g K_g^m(k)$ converge to the diagonal entries of $q_0^{-1} g q_0$ and its below diagonal entries converge to 0.

The QR-algorithm (without shifts). Write an invertible $n \times n$ matrix $A = QR$ where $Q \in K$ and $R \in U'$ and define $A' = RQ$. Hence $A' = Q^{-1}AQ = \pi(A)^{-1} A \pi(A)$. The QR algorithm produces a sequence of matrices A_i,

$$A_0 = A \text{ and for } i > 0,\ A_i = A_{i-1}' = R_{i-1}Q_{i-1}$$

where $A_{i-1} = Q_{i-1}R_{i-1}$ is the QR factorization of A_{i-1}. Thus, inductively,

$$A_i = Q_{i-1}^{-1} \cdots Q_0^{-1}A_0Q_0 \cdots Q_{i-1}, \quad i > 0.$$

Also (see Wilkinson's discussion of Rutishauser's and Francis's work) by simple induction (for $i \geq 0$)

$$A^{i+1} = Q_0 \cdots Q_iR_i \cdots R_0$$

since

$$\begin{aligned} A^{i+2} = AA^{i+1} = AQ_0 \cdots Q_iR_i \cdots R_0 \\ = Q_0 \cdots Q_iA_{i+1}R_i \cdots R_0 \\ = Q_0 \cdots Q_iQ_{i+1}R_{i+1}R_i \cdots R_0. \end{aligned}$$

Hence $Q_0 \cdots Q_m = \pi(A^{m+1})$ which, by definition of K_g, equals $K_A^{m+1}(\mathrm{Id})$ and $A_m = K_A^m(\mathrm{Id})^{-1}AK_A^m(\mathrm{Id})$. Thus we have proved:

Corollary 2 (Francis, Kublanovskaya, Rutishauser, Wilkinson). If A is an $n \times n$ invertible matrix and A_k is the k-th matrix produced by the QR algorithm then $A_k = \pi(A^k)^{-1}A\pi(A^k)$. For A with eigenvalues of distinct moduli, A_k "converges" to upper triangular form as in Corollary 1.

Now we turn to the proof of the theorem.

Notation. L will denote the subgroup of lower triangular matrices. G acts (on the left) on $X = G/U$; so also does each subgroup, H, of G. We will be concerned with X as a (left) U space and as a (left) L space. We will use the usual notation, Hx, for the unique H orbit of X containing x.

For $c \in G$, let $\langle c \rangle$ denote the coset $cU \in X$. Let $\lambda_1, \ldots, \lambda_n$ be g's eigenvalues ordered so that $|\lambda_1| > \cdots > |\lambda_n|$. Let $b \in G$ be such that the j-th column of b is an eigenvector of g with eigenvalue λ_j. Thus $gb = bd$ where $d = \mathrm{Diagonal}(\lambda_1, \ldots, \lambda_n)$. Note that b may be chosen with real entries in the real case since our assumption forces the λ's to be real.

Now $\varphi_g = \varphi_{bdb^{-1}} = \varphi_b\varphi_d\varphi_{b^{-1}} = \varphi_b\varphi_d\varphi_b^{-1}$. *Thus questions concerning the dynamics of* φ_g *are equivalent to the corresponding questions concerning the dynamics of* φ_d. It will be more convenient to deal with φ_d.

Let Σ be the subgroup of G consisting of permutation matrices. Σ has n! elements. Since $\Sigma \cap U = \{e\}$, $\{\langle p\rangle\}_{p\in\Sigma}$ is a set of n! points in $X = G/U$.

Note that $\langle p\rangle$ is a fixed point of φ_d because $\varphi_d(\langle p\rangle) = \langle dp\rangle = \langle pp^{-1}dp\rangle = \langle p\rangle$ since $p^{-1}dp$ is diagonal (its entries are a permutation of those of d) and hence in U.

Lemma 1.

(a) $$L\langle p\rangle \subset W^s(\langle p\rangle),$$

(b) $$U\langle p\rangle \subset W^u(\langle p\rangle).$$

Proof. We first note that for any $g \in G$, $\varphi_d(g\langle p\rangle) = (dgd^{-1})\langle p\rangle$. This is because $\varphi_d(g\langle p\rangle) = \langle dgp\rangle = (dgd^{-1})\varphi_d(\langle p\rangle) = dgd^{-1}\langle p\rangle$ (by earlier remark).

Hence $\varphi_d^m(g\langle p\rangle) = (d^mgd^{-m})(\langle p\rangle)$. But $(d^mgd^{-m})_{ij} = \lambda_i^m g_{ij}(\lambda_j^m)^{-1} = (\lambda_i/\lambda_j)^m g_{ij}$. If $g = l \in L$, $l_{ij} = 0$ for $i < j$ and for $i > j$, $\lim_{m\to\infty}(\lambda_i/\lambda_j)^m l_{ij} = 0$. Hence $\lim_{m\to\infty} d^m l d^{-m} = d' = \mathrm{Diag}(l_{11}, \ldots, l_{nn})$. So $\lim_{m\to\infty}\varphi_d^m(l\langle p\rangle) = \langle d'p\rangle = \langle pp^{-1}d'p\rangle = \langle p\rangle$ since - as above - $p^{-1}d'p$ is diagonal and hence in U. This proves (a); the proof of (b) is similar. □

Lemma 2.

(a) $$X = G/U = \bigcup_{p\in\Sigma} L\langle p\rangle \qquad \text{(disjoint union)}$$

(b) $$X = G/U = \bigcup_{p\in\Sigma} U\langle p\rangle \qquad \text{(disjoint union).}$$

Proof. The disjointness is clear from Lemma 1. We prove (b) first. We need show: $G = \cup_{p\in\Sigma} UpU$. This is a well-known result (Bruhat decomposition); for the reader's convenience we include an elementary proof from Steinberg [St]. Consider $b \in G$. Suppose the i-th and j-th rows of b have precisely the same number of leading 0's ($i < j$). Then by adding a suitable multiple of j-th row to the i-th row we get a matrix b_1 having at least one more leading 0 than b has. Now $b_1 = eb$ for a suitable elementary matrix, e. Since $i < j$, $e \in U$. Thus $b_1 \in Ub$.

Let $b_2 \in Ub$ maximize the total number of leading 0's. The above argument shows that the distinct rows of b_2 have distinct numbers of leading 0's. It follows that these numbers must be (in some order) $0, 1, \ldots, n-1$. So there is a $p \in \Sigma$ such that $pb_2 \in U$. Thus $b \in Up^{-1}U$.

To show (b), argue as above but replace "leading 0's" by "trailing 0's". One gets $b_2 \in Ub$ as above a p in Σ such that $pb_2 \in L$. So b is in $Up^{-1}L$ or $b^{-1} \in LpU$. $\square$

Remark. The following are easy consequences of the above.

(i) $L\langle p\rangle = W^s(\langle p\rangle)$ for p in Σ.

(ii) $U\langle p\rangle = W^u(\langle p\rangle)$ for p in Σ.

(iii) A periodic point of φ_d is a fixed point.

Lemma 3. If $x \in W^s(\langle p_1\rangle) \cap W^u(\langle p_2\rangle)$ then these manifolds intersect transversally at x.

Proof. By (i) above, x is in $L\langle p_1\rangle$ and so $Lx = L\langle p_1\rangle = W^s(\langle p_1\rangle)$. Similarly $Ux = W^u(\langle p_2\rangle)$. Hence we must show that the orbits Lx and Ux are transversal submanifolds of X at x. But L and U are transversal in G at the identity element of G. $\square$

Notation. Let L_1 denote the subgroup of lower triangular matrices having 1's on the diagonal.

Lemma 4 ("linearization" of φ_d). For each $p \in \Sigma$:

(a) The map, $L_1 \ni l_1 \to \langle pl_1\rangle \in X$, defines an equivalence, $\zeta_p\colon L_1 \cong O_p$. Here O_p is an open, dense, φ_d-invariant neighborhood of $\langle p\rangle$ in X.

(b) The composition, $\zeta_p^{-1}\varphi_d\zeta_p$: $L_1 \overset{\cong}{\to} L_1$ is conjugation by d_p where $d_p = p^{-1}dp = \mathrm{Diag}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$ for an appropriate permutation, σ, of $\{1,2, \ldots, n\}$.

(c) The eigenvalues of $T_x(\varphi_d)$: $T_x(X) \to T_x(X)$ are $\{\lambda_{\sigma(i)}/\lambda_{\sigma(j)}\}_{i>j}$ (here $x = \langle p \rangle$). In particular, $\langle p \rangle$ is a hyperbolic fixed point of φ_d.

Proof. As is well-known, LR-factorization (i.e. Gaussian elimination without pivoting [Wi]) yields an equivalence, $L_1 \times U \overset{\cong}{\to} G_0$; here G_0 is an open, dense, subset (the "big cell" [Bo]) of $G = GL(n)$. Hence, for each p, ζ_p is an equivalence between L_1 and an open, dense, subset (christened O_p) of X. Furthermore, $\varphi_d\zeta_p(l_1) = \langle dpl_1 \rangle = \langle pd_pl_1 \rangle = \langle pd_pl_1d_p^{-1} \rangle \in X$ (this last equality because d_p is diagonal and hence in U). The rest is obvious. □

These lemmas readily yield Theorem 1.

Remarks. Indeed they really prove quite a bit more. Note that Lemmas 2 and 4 imply that X is the union of the O_p's; *so,* X *is covered by* n! *co-ordinate systems (each of which covers "almost all" of* X) *in each of which* φ_d *is linear.* Note that the hypothesis that the moduli of the d_i's are distinct is used only to prove hyperbolicity; except for this Lemma 4 holds for *any* diagonal matrix d. These coordinate systems emphasize that (in n! different ways) $X = G/U$ is a compactification of C^l. This suggests that the "change of coordinates" maps, $\zeta'^{-1}_{p_1}\zeta'_{p_2}$, are "nice" - perhaps toroidal. Au contraire, computation shows that they are quite complicated rational mappings; in particular they are not toroidal for $n > 2$. For more details see [V]; the mappings are "universal formulae" intimately associated with Gaussian elimination. We have here a curious phenomenon - n! inequivalent linear representations of C^{*n} which are all "birationally equivalent."

G/U may be considered as the *manifold of flags.* A flag is a *nested* sequence $V_1 \subset V_2 \subset \cdots \subset V_n$ where V_i is a vector subspace of dimension i. If V_i is the subspace generated by the first i columns of the matrix b, the corresponding flag is identical with $\langle b \rangle$. g takes the flag $V_1, \ldots, V_n$ to the flag $gV_1, \ldots, gV_n$. This makes the convergence remark after Theorem 1 simple to see. Alternatively one can compute with the "linearizing" coordinate systems of Lemma 4.

The space of flags can also be used to give a *dynamical proof* of Lemma 2 (modulo the others). A linear map with eigenvalues of distinct moduli induces a Morse-Smale diffeomorphism on the projective space of a vector space. See [SM], [P], [Ba]. The fixed points are eigenspaces and there are no other periodic points. Let g be a diagonal matrix with real entries so that no products of two distinct sets of diagonal entries are equal. Then g induces a Morse-Smale diffeomorphism on the projective space of $\Lambda^k(R^n)$ or $\Lambda^k(C^n)$ and hence is globally convergent to fixed points on G_k, the Grassmannian of k planes in R^n or C^n, which is contained in the projective space of Λ^k (the wedge of k vectors determining the same k plane differ by a determinant). Now the manifold of flags is contained in the product of the G_k's. The only fixed flags are given by permutations of the basis and therefore $\cup_{p \in \Sigma} W^u(\langle p \rangle) = G/U$. Lemmas 3 and 1 show that $L\langle p \rangle$ and $W^s(\langle p \rangle)$ have the same dimension and hence that they are equal. So $\cup_{p \in \Sigma} L\langle p \rangle = G/U$.

As remarked in Lemma 3 the decomposition of $X = G/U$ into stable (unstable) manifolds of φ_d is intimately related to the *Bruhat decomposition* of G. See e.g. pages 346 to 352 of [Bo]. Theorem 1 apparently generalizes to this setting. Although we could not locate it, it would not be surprising to find it already in the literature.

In the complex case all the φ_g's of Theorem 1 are conjugate in Homeo(X).

Proof. By structural stability of Morse-Smale diffeomorphisms it suffices to show that the set of such g's in $G = GL(n)$ is path connected. Since G is path connected and $g = bdb^{-1}$ it suffices to observe that $\Delta = \{d' \in G \mid d'$ is diagonal with $|d'_{ii}| > |d'_{jj}|$ for $i < j\}$ is path connected.

Differential equations. Given a matrix g, e^g lies on the one parameter group e^{tg} and, for suitable g's, any matrix h which conjugates e^g to upper triangular form also conjugates g to upper triangular form since eigenvectors of e^g are eigenvectors of g. If the eigenvalues of g have distinct real parts then Theorem 1 applies to e^g and thus $K_{e^{tg}}(Id)^{-1} g K_{e^{tg}}(Id)$ is an isospectral deformation of g and "converges" to an upper triangular matrix. It is tempting to differentiate this equation and define a differential equation on GL(n) whose solutions tend to converge to the upper triangular matrices. This works.

Since the map π: GL(n) $\to K$ is given by the product formula GL(n) $= K \times U'$ the derivative of π at the identity can be computed from a corresponding decomposition of the Lie algebras. In the real case, a convenient decomposition is (anti-symmetric) + (diagonal + matrices with 0's on or below the diagonal); in the complex case, (skew-Hermitian) + (real diagonal + matrices with 0's on or below the diagonal).

Given the complex matrix g with $g = \begin{bmatrix} d_1 & & & \\ & \cdot & & \mathbf{U} \\ \mathbf{L} & & \cdot & \\ & & & d_n \end{bmatrix}$ then

$$g = \begin{bmatrix} 0 & & & \\ & \cdot & & -\mathbf{L}^* \\ \mathbf{L} & & \cdot & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} d_1 & & & \bigcirc \\ & \cdot & & \\ & & \cdot & \\ \bigcirc & & & d_n \end{bmatrix} + \begin{bmatrix} 0 & & & \\ & \cdot & & \mathbf{U}+\mathbf{L}^* \\ & & \cdot & \\ \bigcirc & & & 0 \end{bmatrix}$$

where $\mathbf{L}^*$ is the "conjugate transpose" of L.

$$\text{Let } L(g) = \begin{bmatrix} 0 & & & \\ & \cdot & & -\mathbf{L}^* \\ \mathbf{L} & & \cdot & \\ & & & 0 \end{bmatrix} + \sqrt{-1} \times \mathrm{im} \begin{bmatrix} d_1 & & & \bigcirc \\ & \cdot & & \\ & & \cdot & \\ \bigcirc & & & d_n \end{bmatrix}.$$

We omit discussion of the real case for two reasons: (i) it is strictly analogous and (ii) it is a special case. Note that in the real case the second term in L(g) is 0.

The discussion yields:

Lemma 5. $T_e(\pi)(g) = L(g)$.

Lemma 6. The flow $\phi_t(k) = K_{e^{tg}}(k)$ solves the differential equation

$$\dot{k} = k \cdot T_e(\pi)(k^{-1}gk) = k \cdot L(k^{-1}gk) \quad \text{on} \quad K.$$

Proof. $\phi_t(k) = \pi(e^{tg}k) = \pi(kk^{-1}e^{tg}k) = k\pi(k^{-1}e^{tg}k)$. So

$$\frac{d}{dt}(\phi_t(k))_{t=0} = kT_e(\pi)(k^{-1}gk). \qquad \square$$

Proposition 1. $\phi_t(g) = \pi(e^{tg})^{-1}g \ \pi(e^{tg})$ solves the differential equation

$$\dot{g} = [g, L(g)] \quad \text{on} \quad GL(n).$$

Proof. $\pi(e^{tg}) = K_{e^{tg}}(e)$. So

$$\frac{d}{dt}\pi(e^{tg})\Big|_{t_0} = \pi(e^{t_0 g})T_e(\pi)(\pi(e^{t_0 g})^{-1}g \ \pi(e^{t_0 g}))$$

and

$$\frac{d}{dt}\phi_t(g)\Big|_{t_0} = -\pi(e^{t_0 g})^{-1}\frac{d}{dt}\pi(e^{tg})\Big|_{t_0}\pi(e^{t_0 g})^{-1}g\pi(e^{t_0 g})$$

$$+ \pi(e^{t_0 g})^{-1}g\frac{d}{dt}\pi(e^{tg})\Big|_{t_0}$$

$$= - T_e(\pi)(\phi_{t_0}(g)) \cdot \phi_{t_0}(g) + \phi_{t_0}(g)T_e(\pi)(\phi_{t_0}(g))$$

$$= [\phi_{t_0}(g), L(\phi_{t_0}(g))]. \qquad \square$$

Theorem 2. *Let* $\dot{g} = [g, L(g)]$ *be the differential equation on* GL(n) *given by the vector field defined by the Lie bracket of* g *and* L(g). *Let* ϕ_t *be its flow. Then*

1) ϕ_t *is isospectral.*

2) *If the eigenvalues of* g *have distinct real parts then* $\phi_t(g)$ *tends to the upper triangular matrices as* $t \to \infty$ *(as in Corollary 1).*
3) *If* g *is a real symmetric or Hermitian matrix then* $\phi_t(g)$ *tends to a diagonal matrix.*

Proof. All that is left to prove is 3). Since a Hermitian or real symmetric matrix g has real eigenvalues the only way for e^{tg} to have eigenvalues of equal modulus is to have multiple eigenvalues. Now the remark after statement of Theorem 1 shows that the diagonal entries converge and that the below diagonal entries tend to 0. The formula shows that the symmetry properties persist and so the above diagonal entries also converge to 0. □

Theorem 2 was motivated for us by Deift, Nanda and Tomei [DNT] but see also the earlier paper [Sy] by Symes. The flow which they consider is restricted to symmetric tri-diagonal matrices. In this case they attribute 1) to Flaschka and 3) to Moser. Flaschka considered this isospectral flow of Lax type as a change of variables of the Toda lattice equations for n particles on a line with exponential nearest neighbor interactions and proved that this system is completely integrable. The Hamiltonian is:

$$\mathscr{H} = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} \exp(x_k - x_{k+1}).$$

Flaschka's change of variables is:

$$a_k = -y_k/2 \qquad k = 1, \ldots, n$$

$$b_k = \frac{1}{2}\exp\left[\frac{1}{2}(x_k - x_{k+1})\right] \qquad k = 1, \ldots, n-1$$

and the Toda differential equations become expressible in matrix form as

$$\frac{dA}{dt} = [A, L(A)],$$

where

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 & & 0 \\ b_1 & a_2 & b_2 & 0 & & \cdot \\ 0 & b_2 & a_3 & b_3 & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & b_{n-1} \\ 0 & \cdot & \cdot & 0 & b_{n-1} & a_n \end{bmatrix}.$$

The isospectral character of the flow gives integrals of the system and the ultimate behavior is used to show that these integrals are in involution. At the end of their paper Deift, Nanda and Tomei comment that Nanda's thesis deals with general symmetric matrices independent of the band size.

References

[Ba] S. Batterson, "Structurally stable Grassmann transformations," *Trans. Amer. Math. Soc.*, vol. 231 (1977), pp. 385-404.

[Bo] A. Borel, *Linear Algebraic Groups*, Benjamin (1969).

[DNT] P. Deift, T. Nanda, and C. Tomei, "Differential equations for the symmetric eigenvalue problem," preprint.

[F] J. G. F. Francis, "The QR Transform I and II," *Comput. J.* 4 (1961), 265-271, (1962), 332-345.

[HPS] M. W. Hirsch, C. C. Pugh, M. Shub, *Invariant manifolds*, LNM #583, Springer-Verlag (1977).

[K] V. N. Kublanovskaya, "On some algorithms for the solution of the complete eigenvalue problem," *USSR Comput. Math. Phys.*, 3 (1961), 637-657.

[P] G. de la Roque Palis, "Linearly induced vector fields and R^2 actions," *J. Diff. Geom.* 13 (1978).

[R] H. Rutishauser, "Une methode pour la determination des valeurs propres d'une matrice," *Comptes Rend. Acad. Sci.*, Paris 240 (1955), 34-36.

"Solution of eigenvalue problems with the LR transformation," *Nat. Bur. Stand. Appl. Math. Ser.* 49 (1958), 47-81.

[Sm] S. Smale, "Differentiable Dynamical Systems," *Bull. Amer. Math. Soc.* 73 (1967), 747-817, MR 37 #3598; erratum, 39, p. 1593. Also available in *The Mathematics of Time*, Springer-Verlag (1980).

[St] R. Steinberg, *Conjugacy classes in algebraic groups*, LNM #366, Springer-Verlag (1974).

[Sy] W. W. Symes, "The QR-algorithm and scattering for the finite nonperiodic Toda lattice," *Physica 4D*, (1982), 275-280.

[V] A. T. Vasquez, "The QR-algorithm and the flag manifold," class notes (Spring '83).

[Wa] J. H. Watkins, "Understanding the QR-Algorithm," *SIAM Rev.*, vol. 24, no. 4 (1982), 427-440.

[Wi] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford Univ. Press (1965).

"Calculation of eigensystems of matrices," in *Numerical Analysis*, Ed. by J. Walsh, Academic Press, London (1966).

IBM, T. J. WATSON RESEARCH CENTER
P. O. BOX 218
YORKTOWN HEIGHTS, NEW YORK 10598

GRADUATE SCHOOL
CITY UNIVERSITY OF NEW YORK
NEW YORK, NEW YORK 10036