Beyond Hyperbolicity

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Beyond hyperbolicity

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The goal here is to study properties of (discrete) dynamical systems not possessing the frequently analyzed properties of structural stability or hyperbolicity (e.g., Axiom A). In particular, we relate the two a priori disparate notions of filtrations and Ω-explosions.

The context is a compact C^∞ manifold M and Diff (M), C^r diffeomorphisms, C^r topology, $0 \leqq r \leqq \infty$, r fixed throughout. Because of Lemma 4, we need $\dim M > 2$, although presumably more work could remove this assumption, at least from the main theorem. We recall that a *filtration* for f is a finite, ordered collection $\{M_\alpha\}$, $\alpha = 1, \cdots, n$, where each M_α is a submanifold with boundary of M, dimension $M_\alpha =$ dimension M, with $\operatorname{int} M_\alpha \supset M_{\alpha-1}$, and $f(M_\alpha) \subset \operatorname{int} M_\alpha$, each α. We also suppose that $M_1 = \phi$, $M_n = M$.

A filtration has two obvious but important properties: (1) stability under even C^0 perturbations, i.e., if $\{M_\alpha\}$ is a filtration for f, then there is a neighborhood $N(f)$ of f in Diff (M), C^0 topology, such that if $g \in N(f)$, then $\{M_\alpha\}$ is a filtration for g; (2) gives a decomposition of the nonwandering set $\Omega = \Omega(f)$. More precisely, recall that Ω is the closed invariant subset of points x of M with property, given any neighborhood U of x, there is $m > 0$ such that $f^m(U) \cap U \neq \phi$. If $\{M_\alpha\}$ is a filtration for f, then let $\Omega_\alpha = (M_\alpha - M_{\alpha-1}) \cap \Omega$. Then the Ω_α give a finite, disjoint decomposition of Ω into compact invariant subsets.

Filtrations exist for any diffeomorphism, e.g., take $M_1 = \phi$, $M_2 = M$. An additional useful property that a filtration might possess and prevent this sort of triviality is as follows.

Let $\Lambda_\alpha = \bigcap_{m \in Z} f^m(M_\alpha - M_{\alpha-1})$. Note that Λ_α is compact and contained in the interior of $M_\alpha - M_{\alpha-1}$. Λ_α is the maximal invariant set for f on $M_\alpha - M_{\alpha-1}$. Clearly, $\Lambda_\alpha \supset \Omega_\alpha$ for each α. We say that $\{M_\alpha\}$ is a *fine* filtration if $\Lambda_\alpha = \Omega_\alpha$ for each α.

We make a detour by stating some problems related to fine filtrations.

Problem (1): Let $f\colon D^n \to D^n$ be a diffeomorphism of the n-disk into itself (even for $n = 2$) such that $\Lambda(f) = \Omega(f)$ is true for f as well as C^r perturbations $\left(\Lambda(f) = \bigcap_{m>0} f^m(D^n)\right)$. Is $\Omega(f)$ a point? Clearly, $\Omega(f)$ has the Čech cohomology of a point.

Problem (2): For a fine filtration $\{M_\alpha\}$ of f, $M_1 = \phi$, $\Omega_1 = \phi$, and we have (Čech theory) $H^*(\Omega_2) = \text{limit}\,\{H^*(M_2) \xrightarrow{f^*} H^*(M_2)\}$. Can one find an extension of this statement for the other Ω_α?

General conditions are known for the existence of fine filtrations. Axiom A and the no-cycle condition (there is no need to recall the definition here) imply the existence of a fine filtration and this is a major step in the proof of the Ω-stability theorem. For a general reference for this whole subject, see the survey [1]. Fine filtrations exist under more general conditions than Axiom A, and it is a subtle question as to whether their existence is a generic property. The answer turns out to be negative by a remarkable, yet unpublished example of S. Newhouse concerning a set of diffeomorphisms of the 2-sphere.

This example motivated us to define a *fine sequence* of filtrations. This is a sequence of filtrations, indexed by $k = 1, 2, \cdots$, $\{M_\alpha^k\}$, $\alpha = 1, \cdots, n_k$, such that

(1) $\{M_\alpha^k\}$ refines $\{M_\beta^{k-1}\}$, each $k > 1$, i.e., for each α, $M_\alpha^k - M_{\alpha-1}^k$ is contained in $M_\beta^{k-1} - M_{\beta-1}^{k-1}$ for some β; and

(2) $\bigcap_{k>0} \Lambda^k = \Omega$, where $\Lambda^k = \bigcup_\alpha \Lambda_\alpha^k$ and Λ_α^k is defined for each k as before.

Then a fine filtration is a fine sequence, constant in k, and it can be shown that Newhouse's example has a fine sequence.

On the other hand, a fine sequence of filtrations gives an approximation of a fine filtration and is the best that can be hoped for in general by Newhouse's example. It isn't known if possession of a fine sequence is a generic property for C^r diffeomorphism.

We say that f in Diff (M) does not permit C^0 Ω-explosions if, given an open neighborhood $U(\Omega(f))$, there is a neighborhood $N(f)$ in Diff (M), C^0 topology such that $\Omega(g) \subset U(\Omega)$, any g in $N(f)$. If f has a fine sequence of filtrations, then it doesn't permit C^0 Ω-explosions. This is seen as follows. We are given a fine sequence for f and also $U(\Omega)$. Choose k such that $U(\Omega) \supset \Lambda^k \supset \Omega$. Now, fix k. For each α, $1 < \alpha \leqq n_k$, we can find m, $q > 0$ such that $\Omega_\alpha \subset \Lambda_\alpha^k \subset f^m(M_\alpha^k) - f^{-q}(M_{\alpha-1}^k) \subset U(\Omega)$. In fact, we may suppose the same m, q work for all α. The last inclusion will be true for g in a sufficiently small C^0 neighborhood of f for all α. This implies the assertion.

The main goal of this paper is to prove the converse, so:

THEOREM. *A diffeomorphism f possesses a fine sequence of filtrations if and only if f does not permit C^0 Ω-explosions.*

Toward proving this theorem, we define an *open decomposition* for f to be a finite number of open sets W_α in M, with disjoint closures and such that $\bigcup_\alpha W_\alpha \supset \Omega$. Define $W_\alpha^u = \{x \in M \mid f^{-m}(x) \in W_\alpha$ for some $m \geqq 0\}$. Then $\bigcup_\alpha W_\alpha^u = M$ and each W_α^u is open. Say that $W_\beta \geqq W_\alpha$ if $W_\alpha \cap W_\beta^u \neq \phi$. An *r-cycle* is a set of W_α with $W_{\alpha_1} \geqq W_{\alpha_2} \geqq \cdots \geqq W_{\alpha_{r+1}}$ for $r > 1$, and a *1-cycle* is a W_α with some $x \notin W_\alpha$, m, $q > 0$, and $f^m(x)$, $f^{-q}(x) \in W_\alpha$. Then $\{W_\alpha\}$ has the no-cycle property if there are no r-cycles, $r > 0$. Let $\Lambda_\alpha = \bigcap_{m \in Z} f^m(W_\alpha)$.

LEMMA 1. *Let $\{W_\alpha\}_{\alpha \in A}$ be an open decomposition for $f \in \mathrm{Diff}\,(M)$ with the no-cycle property and each Λ_α compact. Then there is a filtration $\{M_\alpha\}$ such that $\Lambda_\alpha \subset M_\alpha - M_{\alpha-1}$ for each α.*

Remarks.

(1) The ordering of the filtration will be compatible with the relation $\geqq$ on the W_α.

(2) One can find such an $\{M_\alpha\}$ (perhaps with a bigger indexing set) which refines any given filtration.

For the proof, choose a simple ordering on A compatible with $\geqq$ and let $M'_\alpha =$ the closure of $\bigcup_{\beta \leqq \alpha} W_\beta^u$. This almost does it since $\Lambda_\alpha \subset W_\alpha \subset M'_\alpha - M'_{\alpha-1}$ and $f(M'_\alpha) \subset M'_\alpha$. However, $f(M'_\alpha)$ is not necessarily in the interior of M'_α (int M'_α), and the proof must be a little more elaborate.

We proceed inductively to define M_α as follows. $M_1 = \phi$ and let $N_2 = W_2$. Then $\bigcap_{m \geqq 0} f^m(\bar{N}_2) = \Lambda_2 \subset N_2$. Here $\bar{N}_2$ is Cl N_2, or the closure of N_2, and we have used the no-cycle property. Then, since Λ_2 is compact, (cf. Lemma 4.2 of [2]), there is a compact neighborhood P_2 of Λ_2 contained in N_2 with $f(P_2) \subset \mathrm{int}\,(P_2)$. Finally, choose a compact manifold neighborhood M_2 of P_2 in N_2 with $f(M_2) \subset \mathrm{int}\, M_2$ to complete the first step of the inductive process.

The next step begins by letting $N_3 = M_2 \cup W_3^u$. Then, using the no-cycle property, it follows that $\bigcap_{m>0} f^m(\bar{N}_3) = W^u(\Lambda_3) \cup \Lambda_2$, where

$$W^u(\Lambda_3) = \{x \in M \mid f^m(x) \longrightarrow \Lambda_3 \text{ as } m \longrightarrow -\infty\}$$

(note $W^u(\Lambda_2) = \Lambda_2$). Furthermore, $W^u(\Lambda_3) \cup \Lambda_2$ is compact since $\partial W_3^u \subset \Lambda_2$ (here $\partial W_3^u = \mathrm{Cl}\, W_3^u - W_3^u$). Then M_3 is constructed from N_3 just as in the previous step.

The general induction step proceeds similarly, with $N_k = W_k^u \cup M_{k-1}$, $\bigcap_{m \geqq 0} f^m(\bar{N}_k) = \bigcup_{j \leqq k} W^u(\Lambda_j)$, and $W^u(\Lambda_k) \subset \bigcup_{j<k} W^u(\Lambda_j)$. This proves Lemma 1.

LEMMA 2. *Let $f \in \mathrm{Diff}\,(M)$ not permit any C^0 Ω-explosions and a neighborhood $U(\Omega)$ be given. Then there is an open decomposition $\{W_\alpha\}_{\alpha \in A}$ with the no-cycle property, compact Λ_α and $\bigcup_\alpha W_\alpha \subset U(\Omega)$.*

Lemmas 1 and 2, together with Remark (2) after Lemma 1, yield the theorem.

For the proof of Lemma 2, suppose M has a metric coming from a Riemannian metric and this metric induces a C^0 metric on Diff (M). Choose $\delta > 0$ so that if the C^0 distance from $g \in \text{Diff}(M)$ to f is less than δ, then $\Omega(g) \subset U(\Omega)$.

Choose a finite covering of Ω of open convex balls B_i, each $B_i \subset U(\Omega)$, diameter $B_i < \delta$. Let U_β denote the components of $\cup B_i$. By shrinking the B_i a bit, we may suppose the U_β to have disjoint closure and be finite in number.

Say U_α is equivalent to U_β if there is a common cycle containing U_β and U_α and let W_α be the union of members of an equivalence class. Then $\{W_\alpha\}$ is an open decomposition with no r-cycles, $r > 1$. Let $W_\alpha^s = \{x \in M \mid f^m(x) \in W_\alpha,$ some $m \geqq 0\}$ and $V_\alpha = W_\alpha^u \cap W_\alpha^s$. Then the finite number of V_α are open, $\bigcup_\alpha V_\alpha \supset \Omega$, and $\{V_\alpha\}_\alpha$ has no cycles. Any $x \in \text{Cl}\,\Lambda_\alpha - \Lambda_\alpha$ leads to a 1-cycle since $f^m(x) \in \text{Cl}\,\Lambda_\alpha$, all m. This implies that Λ_α is compact, each α. Finally, one can shrink the V_α a bit if necessary to insure that their closures are disjoint.

It remains to prove that each V_α is contained in $U(\Omega)$.

The idea of the proof is to create an Ω-explosion by taking $x \in V_\alpha - U(\Omega)$ and perturbing f by less than δ to make x a periodic point.

To this end, define a chain of δ balls between y, z for any $f \in \text{Diff}(M)$ to be a sequence $U_1, \cdots, U_n$ of convex open balls of diameter $<\delta$, $y \in U_1$, $z \in U_n$, $U_i \cap \Omega \neq \phi$, and for each $i = 1, \cdots, n-1$, there exists $m \geqq 0$ such that $f^m(U_i) \cap U_{i+1} \neq \phi$.

LEMMA 3. *Given a chain of δ-balls between y, z, there is a $g \in \text{Diff}(M)$, with the C^0 distance between f, g less than δ, $g^N(f^{-1}(y)) = f(z)$, some $N > 0$ and $g = f$ outside $\bigcup_{i=1}^n U_i$.*

Postponing the proof of Lemma 3 for a moment, we see how it finishes the proof of Lemma 2 and hence of the theorem.

Suppose $x \in V_\alpha$, $x \notin U(\Omega)$. Since $x \in W_\alpha^u \cap W_\alpha^s$, there exists m, $q > 0$ such that $y = f^m(x) \in W_\alpha$, $f^{-q}(x) = z \in W_\alpha$. We suppose m, q minimal with this property. Then there is a chain of δ-balls between y and z of the B_i used in constructing W. Application of Lemma 3 yields a g having x as a periodic point, contradicting our choice of δ.

Part of the idea of Lemma 3 is in the following. Here is where the hypothesis dim $M > 2$ is used.

LEMMA 4. *Let (q_i, p_i), $i = 1, \cdots, l$ be pairs of points on a compact manifold M, all disjoint such that $d(q_i, p_i) < \delta$. Then there is a diffeomorphism $\eta: M \to M$ within C^0 distance δ of the identity such that $\eta(q_i) = p_i$ for each i.*

For the proof, one uses disjoint arcs α_i joining q_i to p_i, each i and applies a standard theorem from differential topology obtaining η whose support is in the disjoint cell neighborhoods of each arc α_i.

We proceed to the proof of Lemma 3.

Let $y_i \in f^{n_i}(U_i) \cap U_{i+1}$, $i = 1, \cdots, n-1$, where $n_i \geqq 0$ is the smallest possible with nonempty intersection. Let $z_i = f^{-n_i}(y_i)$ and $w_i \in U_i$, $l_i \geqq 0$ be chosen such that $f^{l_i}(w_i) \in U_i$. This is always possible since $U_i \cap \Omega \neq \phi$. Now by changing f a little, if necessary, we may assume that the points $f^{-1}(y)$, y, z, $f(z)$, y_i, z_i, and $f^j(w_i)$, $0 \leqq j \leqq l_i$, are all distinct.

Our goal is to perturb f to g which maps $y \to w_1 \to z_1 \to w_2 \to z_2 \to \cdots z_{n-1} \to z$ under positive iterates.

For this purpose, we consider the pairs of points $\{(\alpha, \beta)\} = (y_1, w_1)$; $\big(f^j(w_i), f^j(w_i)\big)$ for $0 < j < l_i$; $\big(f^{l_i}(w_i), z_i\big)$; (y_i, w_{i+1}), (w_i, z), and finally, $\big(f(z), f(z)\big)$. By Lemma 4, there is a diffeomorphism η with support in $\bigcup_{i=1,\cdots,n} U_i$ so that $\eta(\alpha) = \beta$, η has C^0 size less than δ and the support of $\eta \subset \bigcup_{i=1,\cdots,n} U_i$. $\eta \circ f$ is the diffeomorphism we were seeking.

$$\eta \circ f\big(f^{-1}(y)\big) = w_1; \quad \eta \circ f(w_1) = f(w_1);$$

and

$$\eta \circ f(f^{l_i - 1} w_1) = z_1; \quad \eta \circ f(z_1) = w_2;$$

etc.

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